Local scaling limits of Lévy driven fractional random fields

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To the memory of Vladas

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1. $\gamma$-tangent and $\gamma$-rectangent local scaling limits and scaling transition
2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples
3. Main results
4. Extensions and comments

V. Pilipauskaitė, D. Surgailis (2021) Local scaling limits of Lévy driven fractional random fields, Preprint
http://arxiv.org/abs/2102.00732
1. $\gamma$-tangent and $\gamma$-rectangent RFs and scaling transition

$X = \{X(t), \; t \in \mathbb{R}^2\}$: a random field (RF), $t_0 = (t_{01}, t_{02}) \in \mathbb{R}^2$: given point

Two types of increment of $X$ at $t_0$:

*(ordinary) increment*: $X(t_0 + t) - X(t_0)$, and

*rectangular increment*:

$X[t_0, t_0 + t] := X(t_{01} + t_1, t_{02} + t_2) - X(t_{01}, t_{02} + t_2) - X(t_{01} + t_1, t_{02}) + X(t_{01}, t_{02})$
1. $\gamma$-tangent and $\gamma$-rectangent RFs

- both types of increments give rise to *increment RF of $X$ at $t_0$* (indexed by $t \in \mathbb{R}^2$)
- RF $X$ with stationary (ordinary) increments or with stationary rectangular increments (different notions)
- this talk: local (small scale) scaling limits of both types of increment RFs as $t \to 0$ for a class of ‘fractional’ RF $X$ on $\mathbb{R}^2$
- important to *infill statistics* of RFs
- the scaling limits (*tangent RFs*) depend on how $t = (t_1, t_2)$ tends to $0 = (0, 0)$:
  \[ t_1 = \lambda x_1, \quad t_2 = \lambda^\gamma x_2 \]
  where $\lambda \to 0$ and $\gamma > 0$ is fixed
- $\gamma > 0$ characterizes scaling anisotropy ($\gamma = 1$: isotropic scaling)
Γ := diag(1, γ), λ^Γ = diag(1, λ^γ), λ^Γ t = (λ t_1, λ^γ t_2) ∈ ℝ^2

**Definition**

Suppose there exist normalization \( d_{\lambda,\gamma} \downarrow 0 \) (\( \lambda \downarrow 0 \)) s.t.

\[
d_{\lambda,\gamma}^{-1}(X(t_0 + \lambda^\Gamma t) - X(t_0)) \xrightarrow{\text{fdd}} T_\gamma(t),
\]

\[
d_{\lambda,\gamma}^{-1}X[t_0, t_0 + \lambda^\Gamma t] \xrightarrow{\text{fdd}} V_\gamma(t),
\]

\( T_\gamma \) and \( V_\gamma \) in (1), (2) are called \( \gamma \)-tangent and \( \gamma \)-rectangent RFs of RF \( X \) at \( t_0 \) respectively.

- 1-tangent or tangent (isotropic scaling) RF \( T_1 \) in (1) was introduced in Falconer (2002) (\( \xrightarrow{\text{fdd}} \) replaced by a functional convergence)
- generalizes the concept of tangent process for \( X = \{X(t)\} \) with \( t ∈ ℝ \)
- normalization \( d_{\lambda,\gamma} \downarrow 0 \) generally different for (1) and (2)
- dependence on \( t_0 \) on r.h.s. of (1) and (2) is suppressed (do not depend on \( t_0 \) by stationarity of increments in this talk)
- ‘rectangent’ = abridge for ‘rectangular tangent’
1. $\gamma$-tangent and $\gamma$-rectangent RFs

- Since $T_1$ is self-similar (SS) the existence of $T_1$ in (1) also termed ‘local asymptotic self-similarity’ (Benassi, Cohen, Istas (2004), Cohen (2012), Cohen, Istas (2013))
- The above papers proved the existence of $T_1$ for a class of isotropic fractional Lévy RFs on $\mathbb{R}^d$
- Under mild conditions all scaling limits in (1)–(2) satisfy the $(H, \gamma)$-SS property:

$$U(\lambda^{\gamma} t) \overset{\text{fdd}}{=} \lambda^H U(t), \quad \forall \lambda > 0,$$

with some $H = H(\gamma) > 0$, normalization $d_{\lambda, \gamma}$ is $H$-regularly varying as $\lambda \downarrow 0$

**Definition**

Suppose $\gamma$-rectangent limits in (2) exist for any $\gamma > 0$. We say that these limits exhibit scaling transition at some $\gamma_0 > 0$ if

$$V_\gamma = \begin{cases} 
V_+, & \gamma > \gamma_0, \\
V-, & \gamma < \gamma_0, \\
V_0, & \gamma = \gamma_0
\end{cases} \quad \text{and} \quad V_+ \overset{\text{fdd}}{\neq} aV_- \ (\forall a > 0) \quad (4)$$
1. γ-tangent and γ-rectangent RFs

- Analogous definition in γ-tangent case

- Closely related to large-scale scaling transition for RFs on $\mathbb{Z}^2$ or $\mathbb{R}^2$:
  
  Puplinskaitė, Surgailis (2016). Bernoulli 22, 2401–2441;

- In the large-scale case, rectangular increments are replaced by integrals or sums of a stationary RF $X$ over large rectangle $[0, \lambda x_1] \times [0, \lambda \gamma x_2]$ and one is interested in the limit

  \[
  d_{\lambda, \gamma}^{-1} \int_{[0, \lambda x_1] \times [0, \lambda \gamma x_2]} X(t) \, dt \xrightarrow{\text{fdd}} V_\gamma(x), \quad \lambda \to \infty \tag{5}
  \]

  for any given $\gamma > 0$

- In the above works a similar trichotomy to (2) was observed in large-scale anisotropic scaling for several classes of linear and nonlinear long-range dependent (LRD) RF models The trichotomy was called the scaling transition, with $V_\pm$ the unbalanced and $V_0$ the well-balanced scaling limits.

- Intrinsically related to LRD
1. $\gamma$-tangent and $\gamma$-rectangent RFs

- Large-scale trichotomy or scaling transition of different nature occurs in applied sciences (telecommunications and econometrics) in joint temporal-spatial aggregation of independent LRD processes:

$$d_{\lambda,\gamma}^{-1} \sum_{i=1}^{[\lambda \gamma x_2]} \int_0^{\lambda x_1} (X_i(t) - E X_i(t)) \, dt \xrightarrow{fdd} V_{\gamma}(x), \quad \lambda \to \infty$$

(6)

where $X_i = \{X_i(t), t \in \mathbb{R}\}$ are independent copies of a stationary process $X = \{X(t), t \in \mathbb{R}\}$. Typical examples of $X$:

- ON/OFF process with heavy tailed ON or OFF intervals (telecommunications)
- random-coefficient AR(1) process with random coefficient having a power-law distribution near the unit root (econometrics)

Gaussian or stable limits in (6) depending on whether $\gamma > \gamma_0$ or $\gamma < \gamma_0$

Gaigalas, Kaj (2003). Bernoulli 9, 671–703;
2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

We consider RF $X = \{X(t), t = (t_1, t_2) \in \mathbb{R}_+^2 \}$ on $\mathbb{R}_+^2 = (0, \infty)^2$ written as stochastic integral:

$$X(t) := \int_{\mathbb{R}^2} \left\{ g(t - u) - g_1^0((t_1, 0) - u) - g_2^0((0, t_2) - u) + g_{12}^0(-u) \right\} M(du), \quad (7)$$

w.r.t. Lévy random measure $M(A), A \subset \mathbb{R}_+^2$, with independent values on disjoint sets and characteristic function

$$Ee^{i\theta M(A)} = \exp \left\{ \text{Leb}(A) \left( -\frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta \ell_\alpha(y)) \nu(dy) \right) \right\}, \quad \theta \in \mathbb{R}, \quad (8)$$

where:

- $\sigma^2 \geq 0, \ 0 < \alpha \leq 2$ and $\nu$ is a Lévy measure on $\mathbb{R}$ satisfying Assumption (M)$_\alpha$ below (meaning roughly that $\lambda^{-2/\alpha} M(\lambda A)$ tends to $\alpha$-stable random measure $W_\alpha(A)$ as $\lambda \to 0$)

- $g, g_1^0, g_2^0, g_{12}^0$ are deterministic functions satisfying a power-law behavior at the origin $0 \in \mathbb{R}^2$ (specified in Assumption (G)$_\alpha$)

- centering $\ell_\alpha(y) := y \ (1 < \alpha \leq 2), := 0 \ (0 < \alpha < 1)$
2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

- $g_1^0, g_2^0, g_{12}^0$ ‘initial functions’ do not enter rectangular increment:
  \[
  X]0, t] = \int_{\mathbb{R}^2} g] - u, t - u]M(du),
  \]
  where
  \[
  g] - u, t - u] = g(t_1 - u_1, t_2 - u_2) - g(-u_1, t_2 - u_2) - g(t_1 - u_1, -u_2) + g(-u_1, -u_2)
  \]

- $X$ in (7): stationary rectangular increments: $X]t_0, t_0 + t] \overset{fdd}{=} X]0, t]$

- $X$ in (7): direct 2-dim analog of Lévy driven moving average with 1-dim time:
  \[
  X(t) = \int_{-\infty}^{t} (g(t - u) - g^0(-u))M(du), \quad t \geq 0
  \]

  with kernel $g(t) \sim ct^q$, $t \downarrow 0$ [Basse-O’Connor, Lachièze-Rey, Podolskij (2017) Ann. Probab. 45, 4477–4528]

- [Cohen, Istas (2013) Fractional Fields and Applications. Mathématiques et Applications 73, Springer]: 1-tangent limits of isotropic fractional Lévy RFs on $\mathbb{R}^d$:

  \[
  X(t) = \int_{\mathbb{R}^d} \{ \| t - u \|^H - \| u \|^H \} M(du)
  \]

  (particular parametric case of (7))

- we consider anisotropic power behavior of $g(t)$ at $t = 0 \in \mathbb{R}^2$ characterized by two exponents $q_1, q_2 > 0$
2. Lévy driven fractional RFs on \( \mathbb{R}^2 \). Examples

\[
g(t) \sim g_0(t) := \rho(t)^\chi L(t), \quad t \to 0,
\]

where:

- \( \rho(t) := |t_1|^{q_1} + |t_2|^{q_2} \) (anisotropic radial generalized invariant function)
- \( q_1 > 0, q_2 > 0, \chi \neq 0 \): parameters, \( Q := \frac{1}{q_1} + \frac{1}{q_2} \)
- \( L(t), t \in \mathbb{R}^2_0 \) (angular generalized invariant function); some regularity conditions
- \( \chi > 0 (\chi < 0) \): (9) vanishes (explodes) at \( t = 0 \)
- rectangent limits of RF \( X \) depend on two parameters only:

\[
p_i := q_i(Q - \chi) > 0, \quad i = 1, 2,
\]

Example (Fractional Lévy RF)

\[
X(t) = \int_{\mathbb{R}^2} \{ \|t - u\|^{H-\frac{2}{\alpha}} - \|u\|^{H-\frac{2}{\alpha}} \} M(du)
\]
2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

- If $EM(du)^2 = \sigma^2 du$, $\alpha = 2$, $0 < H < 1$ then
  
  $$EX(t)X(s) = E|X(e_1)|^2 \frac{1}{2}(\|t\|^{2H} + \|s\|^{2H} - \|t - s\|^{2H}), \quad t, s \in \mathbb{R}^2, \quad (11)$$


- Satisfies fractional PDE with Laplace operator (particular case of Ex 2)
- Satisfies (9) with $g_0 = g$, $q_1 = q_2 = 2$, $Q = 1$, $\chi = \frac{H}{2} - \frac{1}{\alpha} \in (-\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{\alpha})$.

Example (isotropic Matérn RF)

$$X(t) = \begin{cases} Y(t) - Y(0), & \chi > 0, \\
Y(t), & \chi < 0, \end{cases} \quad (12)$$

$$\left(c^2 - \Delta\right)^{1+\chi} Y(t) = \dot{M}(t) \quad \text{(fractional PDE, } \Delta = \text{Laplace)} \quad (13)$$
2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

- If $EM(du)^2 = \sigma^2 du$ and $\chi > -\frac{1}{2}$, then $E|Y(0)|^2 < \infty$ and

$$EY(0)Y(t) = E|Y(0)|^2 \left( \frac{(c\|t\|)^{1+2\chi} K_{1+2\chi}(c\|t\|)}{\Gamma(1+2\chi)2^{2\chi}} \right)$$

($K_\nu$ : modif. Bessel function)

- Matérn RFs and covariance functions widely used in spatial applications [Guttorp, Gneiting (2006). *Biometrika* 93, 989–995]

- Satisfies (9) with $g_0(t) = \|t\|^{2\chi}$, $q_1 = q_2 = 2$, $Q = 1$, $\chi \in (-\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{\alpha})$

### Example (anisotropic heat operator RF)

\[ (c_1 + \Delta_{12})^{\chi + \frac{3}{2}} X(t) = \dot{M}(t), \quad t \in \mathbb{R}^2, \]

\[ \Delta_{12} := \frac{\partial}{\partial t_1} - c_2^2 \frac{\partial^2}{\partial t_2^2} \quad \text{(heat operator)} \]

- Fundamental solution of (14) (seems new?):

\[ g(t) = \frac{t_1^\chi}{2^{\frac{3}{2}}(2\pi)^{\frac{3}{2}} c_2 \Gamma(\chi + \frac{3}{2})} \exp \left\{ -c_1 t_1 - \frac{t_2^2}{4c_2^2 t_1} \right\}, \quad t_1 > 0 \]
3. Main results


- Satisfies (9) with \( g_0(t) := \rho(t)^\chi \ell(t) \), \( \rho(t) := |t_1| + |t_2|^2 \), \( q_1 := 1 \), \( q_2 := 2 \), \( Q = \frac{3}{2} \), and continuous angular function

\[
\ell(t) := \frac{z^\chi}{2^{\frac{1}{2}}(2\pi)^{\frac{3}{2}} c_2 ^\Gamma(\chi + \frac{3}{2})} \exp \left\{ -\frac{1}{4c_2^2} \left( \frac{1}{z} - 1 \right) \right\}, \quad t_1 > 0, \quad z := \frac{t_1}{\rho(t)} \in (0, 1].
\]

3. Main results [\( \gamma \)-rectangent limits of Lévy driven RFs]


A RF \( V = \{ V(t), t \in \mathbb{R}^2_+ \} \) is said \((H_1, H_2)\)-multi-self-similar (MSS) with parameters \( H_i \geq 0, \ i = 1, 2 \) if

\[
V(\lambda_1 t_1, \lambda_2 t_2) \overset{\text{fdd}}{=} \lambda_1^{H_1} \lambda_2^{H_2} V(t), \quad \forall \lambda_1 > 0, \ \forall \lambda_2 > 0.
\]
3. Main results

Classical example of MSS RF: *Fractional Brownian Sheet (FBS) $B_{H_1,H_2}$*: Gaussian process on $\mathbb{R}_+^2$ with zero mean and covariance

$$
\mathbb{E}B_{H_1,H_2}(t)B_{H_1,H_2}(s) = (1/4)\prod_{i=1}^{2}(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}), \quad (17)
$$

$t = (t_1, t_2) \in \mathbb{R}_+^2$, $s = (s_1, s_2) \in \mathbb{R}_+^2$

- Usually $B_{H_1,H_2}$ is defined for $H_i \in (0,1]$ or $H_i \in (0,1)$
- We extend $B_{H_1,H_2}$ to $H_i \in [0,1]$ by continuity in (17)
- Extension to $H_1 \land H_2 = 0$ leads to very unusual and extremely singular RF (non-measurable paths!)
- Similarly by continuity define FBM $B_0$ with $H = 0$ as Gaussian process on $\mathbb{R}_+$ with zero mean and covariance $\mathbb{E}B_0(t)B_0(s) = 1 - \frac{1}{2}I(t \neq s)$
- $B_0$ is self-similar with $H = 0$ and can be represented as $B_0 \overset{\text{fdd}}{=} \left\{ \frac{1}{\sqrt{2}}(W(t) - W(0)), \; t \in \mathbb{R}_+ \right\}$, where $W(t)$, $t \in [0, \infty)$, is (uncountable) family of independent $N(0,1)$ r.v.s.
3. Main results

Summary of main results: For Lévy driven RFs in (7) with kernel $g$ satisfying (9)

$g(t) \sim \rho(t)^\chi L(t)$, $t \to 0$, $\rho(t) = |t_1|^{q_1} + |t_2|^{q_2}$ and random measure $M \sim \mathcal{W}_\alpha, 0 < \alpha \leq 2$

(i) $\gamma$-rectangent limits $V_\gamma$ exist for any $\gamma > 0$ and are $\alpha$-stable RFs,

(ii) limit family $\{V_\gamma, \gamma > 0\}$ exhibits scaling transition at $\gamma_0 := \frac{q_1}{q_2}$,

(iii) unbalanced limits $V_\pm$ are $(H_1, H_2)$-MSS with one of $H_i$, $i = 1, 2$ equal 1 or 0.

$\gamma$-tangent limits (ordinary increments): more straightforward

Conclusions:

- Gaussian unbalanced limits $V_\pm$ agree with $B_{H_1, H_2}$ with $H_1 \wedge H_2 = 0$ or $H_1 \vee H_2 = 1$
- Unbalanced scaling ($\gamma \neq \gamma_0$) degenerates dependence in one direction (either vertical or horizontal)
- Critical $\gamma_0$ agrees with ‘intrinsic local dependence ratio’ $\frac{q_1}{q_2}$ of RF $X$

Q. How $H_i, i = 1, 2$ depend on $q_1, q_2, \chi, \alpha$? Parameters $p_1, p_2, \alpha, p_i = q_i(Q - \chi) > 0, i = 1, 2, Q = \frac{1}{q_1} + \frac{1}{q_2}$
### 3. Main results

#### Parameter region $V_+$ Hurst parameters $V_-$ Hurst parameters

| $R_{11}$ | $\tilde{\gamma}_{\alpha,1}$ | $0 < H_1 < 1, H_2 = 1$ | $\tilde{\gamma}_{\alpha,2}$ | $H_1 = 1, 0 < H_2 < 1$ |
| $R_{12}$ | $\tilde{\gamma}_{\alpha,1}$ | $0 < H_1 < 1, H_2 = 1$ | $\gamma_{\alpha,1}$ | $0 < H_1 < 1, H_2 = 0$ |
| $R_{21}$ | $\gamma_{\alpha,2}$ | $H_1 = 0, 0 < H_2 < 1$ | $\tilde{\gamma}_{\alpha,2}$ | $H_1 = 1, 0 < H_2 < 1$ |
| $R_{22}$ | $\gamma_{\alpha,2}$ | $H_1 = 0, 0 < H_2 < 1$ | $\gamma_{\alpha,1}$ | $0 < H_1 < 1, H_2 = 0$ |

1 lentelė: Unbalanced rectangent scaling limits $V_\pm$ and their Hurst parameters in regions $R_{ij}, i, j = 1, 2.$
3. Main results

- Four lines in Fig correspond to
  \[ P_{1,1} = \alpha, \quad \frac{P_{1,1}}{\alpha} = \frac{\alpha}{1 + \alpha}, \quad \frac{P_{1,1}}{\alpha}, \frac{1 + \alpha}{\alpha} = 1, \quad \frac{P_{1+\alpha}}{\alpha}, \frac{1}{\alpha} = 1 \]

\[ P_{c_1,c_2} := \frac{c_1}{p_1} + \frac{c_2}{p_2} \]

- \( \alpha \)-stable RFs \( \Upsilon_{\alpha,i}, i = 1, 2 \) with \( H_1 \land H_2 = 0 \) and \( \tilde{\Upsilon}_{\alpha,i}, i = 1, 2 \) with \( H_1 \lor H_2 = 1 \) defined through self-similar \( \alpha \)-stable processes \( Y_{\alpha,i}, \tilde{Y}_{\alpha,i}, i = 1, 2 \) with 1-dim time, particularly,

\[ \tilde{\Upsilon}_{\alpha,1}(t) := t_2 \tilde{Y}_{\alpha,1}(t_1), \quad \tilde{\Upsilon}_{\alpha,2}(t) := t_1 \tilde{Y}_{\alpha,2}(t_2). \]

- Definition of \( \Upsilon_{\alpha,i} \) through \( Y_{\alpha,i} \) more involved (only FDD as probability measure on \( \mathbb{R}^{\mathbb{R}^2_+} \) using Kolmorogov’s consistency theorem)

- Hurst indices:

\[ H_{\alpha,1} := \frac{1 + \alpha}{\alpha} \left( 1 + \frac{p_1}{p_2} \right) - p_1, \quad \tilde{H}_{\alpha,1} := \frac{1 + \alpha}{\alpha} + \frac{p_1}{\alpha p_2} - p_1, \]
\[ H_{\alpha,2} := \frac{1 + \alpha}{\alpha} \left( 1 + \frac{p_2}{p_1} \right) - p_2, \quad \tilde{H}_{\alpha,2} := \frac{1 + \alpha}{\alpha} + \frac{p_2}{\alpha p_1} - p_2. \]
3. Main results (rigorous formulations)

\[ \partial_i f(t) := \partial f(t)/\partial t_i, \quad i = 1, 2, \quad \partial_{12} f(t) := \partial^2 f(t)/\partial t_1 \partial t_2 \]

\( f : \mathbb{R}_0^2 \to \mathbb{R} \) is generalized homogeneous (resp., generalized invariant) if \( \exists \ q_i > 0, \ i = 1, 2 \)

s. t. \( \lambda (\lambda^{1/q_1} t_1, \lambda^{1/q_2} t_2) = f(t) \) \( \forall \ \lambda > 0, \ \forall \ t \in \mathbb{R}_0^2 \) (resp., \( f(\lambda^{1/q_1} t_1, \lambda^{1/q_2} t_2) \) does not depend on \( \lambda > 0 \) \( \forall \ t \in \mathbb{R}_0^2 \)).

Gen. homog. function \( f(t) \) can be represented as \( f(t) = \rho(t)^{-1} \ell(t) \) with

\( \rho(t) = |t_1|^{q_1} + |t_2|^{q_2} \) and a gen. inv. \( \ell(t) = \tilde{\ell}(t_1/\rho(t)^{1/q_1}, t_2/\rho(t)^{1/q_2}) \) where \( \tilde{\ell} \) is restriction of \( f \) to \( \{ t \in \mathbb{R}_0^2 : \rho(t) = 1 \} \)

**Assumptions on kernels** \( g, g_i^0, g_{12}^0 \) of Lévy driven RF \( X \).

**Assumption (G)\( _\alpha \).**

- \( g_0(t) = \rho(t)^\chi L(t) \), where \( L(t) \) is a gen. inv. and \( \chi \in \mathbb{R}_0, \ q_i > 0, \ i = 1, 2, \) with \( Q = \frac{1}{q_1} + \frac{1}{q_2} \) s. t.

\[ -\frac{1}{\alpha} Q < \chi < \left(1 - \frac{1}{\alpha}\right) Q \]

- As \( |t| \to 0, \ g(t) = g_0(t) + o(\rho(t)^\chi), \ \partial_i g(t) = \partial_i g_0(t) + o(\rho(t)^\chi - \frac{1}{q_i}), \ i = 1, 2, \ \partial_{12} g(t) = \partial_{12} g_0(t) + o(\rho(t)^\chi - Q) \) and \( \forall \ t \in \mathbb{R}_0^2, \)

\[ |g_0(t)| \leq C \rho(t)^\chi, \quad |\partial_i g_0(t)| \leq C \rho(t)^\chi - \frac{1}{q_i}, \ i = 1, 2, \quad |\partial_{12} g_0(t)| \leq C \rho(t)^\chi - Q. \]
3. Main results (rigorous formulations)

**Assumption (G)\(_\alpha\).** For any \( t = (t_1, t_2) \in \mathbb{R}^2, \delta > 0, \)
\[
\int_{\mathbb{R}^2} |g(t - u) - g_1^0((t_1, 0) - u) - g_2^0((0, t_2) - u) + g_{12}^0(-u)|^\alpha \, du < \infty \quad (0 < \alpha \leq 2),
\]
\[
\int_{|u| > \delta} \left( \sum_{i=1}^{2} |\partial_i g(u)|^\alpha + |\partial_{12} g(u)|^\alpha \right) \, du < \infty \quad (1 \leq \alpha \leq 2). \tag{18}
\]

Moreover, if \( 0 < \alpha < 1, \) there exist dominating functions \( \bar{g}_i(u), \)
\( \bar{g}_{12}(u), u = (u_1, u_2) \in \mathbb{R}^2_+ \) monotone decreasing in each \( u_i > 0, i = 1, 2, \)
\[ |\partial_i g(u)| \leq \bar{g}_i(|u_1|, |u_2|), |\partial_{12} g(u)| \leq \bar{g}_{12}(|u_1|, |u_2|), |u| > \delta, \]
satisfying (18) with \( \partial_i g, \partial_{12} g \) replaced by \( \bar{g}_i, \bar{g}_{12}. \)

**Assumptions on Lévy random measure \( M \) (characteristics \( (\sigma, \nu) \))**

**Assumption (M)\(_\alpha\).**

- \( \alpha = 2, \sigma > 0 \) and \( \int_{\mathbb{R}} y^2 \nu(dy) < \infty, \) or
- \( 0 < \alpha < 2, \sigma = 0 \) and \( \lim_{y \downarrow 0} y^\alpha \nu([y, \infty)) = c_+, \lim_{y \downarrow 0} y^\alpha \nu((\infty, -y]) = c_- \) for some \( c_\pm \geq 0, c_+ + c_- > 0, \) \( \sup_{y > 0} y^\alpha \nu(\{u \in \mathbb{R} : |u| > y\}) < \infty. \) Moreover, if \( \alpha = 1 \) then \( \nu(dy) = \nu(-dy), y > 0 \) is symmetric.
3. Main results (rigorous formulations)

Assumption \((M)_\alpha\) implies that rescaled Lévy Sheet \(M(t) = \int_{[0,t]} M(du)\) tends to \(\alpha\)-stable Sheet \(W_\alpha(t) = \int_{[0,t]} W(du)\):

\[
(\lambda_1 \lambda_2)^{-\frac{1}{\alpha}} M(\lambda_1 t_1, \lambda_2 t_2) \xrightarrow{\text{fdd}} W_\alpha(t), \quad \lambda_i \downarrow 0, \quad i = 1, 2,
\]

**Theorem**

Let Lévy driven fractional RF \(X\) in (7) satisfy Assumptions \((G)^0_\alpha\), \((G)_\alpha\) and \((M)_\alpha\); \(0 < \alpha \leq 2\), \(\frac{\alpha}{1+\alpha} < P < \alpha\), \(P \neq 1\), \(P \frac{1}{\alpha}, \frac{1+\alpha}{\alpha} \neq 1\), \(P \frac{1+\alpha}{\alpha}, \frac{1}{\alpha} \neq 1\). Then the \(\gamma\)-rectangent RF in (2) exists for any \(\gamma > 0\), \(t_0 \in \mathbb{R}^2_+\) and satisfies the trichotomy

\[
V_\gamma = \begin{cases} V_+, & \gamma > \gamma_0, \\ V_-, & \gamma < \gamma_0, \\ V_0, & \gamma = \gamma_0, \end{cases}
\]

(19)

with \(\gamma_0 = \frac{q_1}{q_2} = \frac{p_1}{p_2}\), \(V_0(t) := \int_{\mathbb{R}^2} g_0] - u, t - u]W_\alpha(du), \), and

\[
V_- := \begin{cases} \tilde{\gamma}_{\alpha,2}, & P \frac{1}{\alpha}, \frac{1+\alpha}{\alpha} > 1, \\ \gamma_{\alpha,1}, & P \frac{1}{\alpha}, \frac{1+\alpha}{\alpha} < 1, \end{cases} \quad V_+ := \begin{cases} \tilde{\gamma}_{\alpha,1}, & P \frac{1+\alpha}{\alpha}, \frac{1}{\alpha} > 1, \\ \gamma_{\alpha,2}, & P \frac{1+\alpha}{\alpha}, \frac{1}{\alpha} < 1. \end{cases}
\]

(20)
4. Extensions and comments

1. What are $\gamma$-rectantgent limits of $X$ when $P = \frac{1}{p_1} + \frac{1}{p_2} > \alpha$ (‘smooth’ kernel $g(t)$)?

Under ‘some’ conditions for any $\gamma > 0$

$$\lambda^{-1-\gamma} X[0, \lambda^\Gamma t] \xrightarrow{\text{fdd}} t_1 t_2 V,$$

where $V := \int_{\mathbb{R}^2} \partial_{12} g(u) M(du)$

(no scaling transition)

2. When $\gamma$-rectantgent limits of $X$ agree with $\alpha$-stable sheet $W_\alpha$?

Assumption $(G)_\alpha$ should be replaced by assuming that $g(t)$ is discontinuous at $t = 0$ and exist ‘limites quadrantales’ $g_{ij} := \lim_{|t| \to 0, t \in \mathbb{R}^2} g(t)$, $i, j \in \{1, -1\}$ on each quadrant $\mathbb{R}_{ij}^2 := \{t \in \mathbb{R}^2 : \text{sgn}(t_1) = i, \text{sgn}(t_2) = j\}$, $i, j = \pm 1$ with

$$g[0] := \sum_{i,j \in \{1,-1\}} ij g_{ij} \neq 0.$$

(no scaling transition)

3. What are $\gamma$-tangent limits (ordinary increments) of $X$?

For RF $X(t) = \int_{\mathbb{R}^2} \left\{ g(t - u) - g_{12}^{0}(-u) \right\} M(du)$ with stationary increments and some related conditions on $g(t)$ and $M$ we prove that $\gamma$-tangent limits $T_\gamma$ in (1) of $X$ exist for any $\gamma > 0$ and
4. Extensions and comments

\[ T_\gamma = \begin{cases} 
T_+, & \gamma > \gamma_0, \\
T-, & \gamma < \gamma_0, \\
T_0, & \gamma = \gamma_0,
\end{cases} \]

where \( \gamma_0 = \frac{q_1}{q_2} \), \( T_0(t) := \int_{\mathbb{R}^2} \left\{ g_0(t - u) - g_0(-u) \right\} W_\alpha(du) \) and
\( T_+(t) := T_0(t_1, 0), \ T_-(t) := T_0(0, t_2) \) depend on only one coordinate on the plane.


5. Functional convergence instead of FDD: open. However, if \( H_1 \land H_2 = 0 \) does not seem feasible.

6. Applications to statistical estimation of \( H_1, H_2 \) from dense rectangular grid: open and challenging problem.