Local scaling limits of Lévy driven fractional random fields

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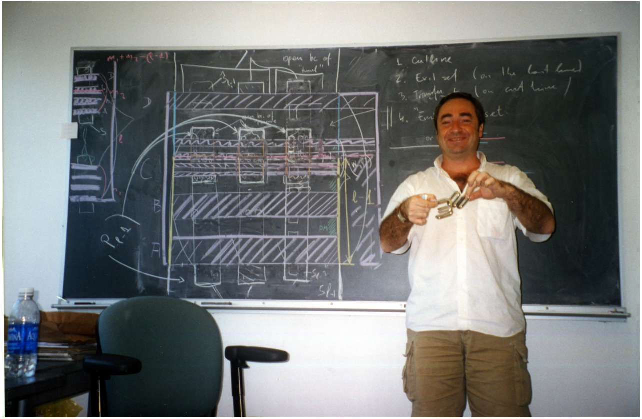
To the memory of Vladas



Cornell, 2003

Outline:

1. γ-tangent and γ-rectangent local scaling limits and scaling transition
2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples
3. Main results
4. Extensions and comments

V. Pilipauskaitė, D. Surgailis (2021) Local scaling limits of Lévy driven fractional random fields, Preprint http://arxiv.org/abs/2102.00732

1. γ-tangent and γ-rectangent RFs and scaling transition

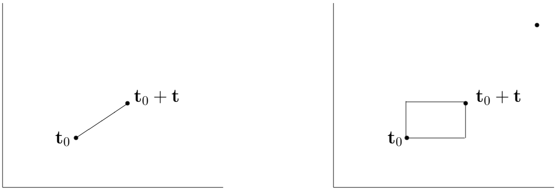
$X = \{X(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{R}^2\}$: a random field (RF), $\boldsymbol{t}_0 = (t_{01}, t_{02}) \in \mathbb{R}^2$: given point

Two types of increment of X at $\boldsymbol{t}_0$:

(ordinary) increment: $X(\boldsymbol{t}_0 + \boldsymbol{t}) - X(\boldsymbol{t}_0)$, and

rectangular increment:

$$X]\boldsymbol{t}_0, \boldsymbol{t}_0 + \boldsymbol{t}] := X(t_{01} + t_1, t_{02} + t_2) - X(t_{01}, t_{02} + t_2) - X(t_{01} + t_1, t_{02}) + X(t_{01}, t_{02})$$



ordinary increment — rectangular increment

1. γ-tangent and γ-rectangent RFs

- both types of increments give rise to *increment RF of X at $\boldsymbol{t}_0$* (indexed by $\boldsymbol{t} \in \mathbb{R}^2$)
- RF X with stationary (ordinary) increments or with stationary rectangular increments (different notions)
- this talk: local (small scale) scaling limits of both types of increment RFs as $\boldsymbol{t} \to \mathbf{0}$ for a class of 'fractional' RF X on $\mathbb{R}^2$
- important to *infill statistics* of RFs
- the scaling limits (*tangent RFs*) depend on *how* $\boldsymbol{t} = (t_1, t_2)$ tends to $\mathbf{0} = (0, 0)$:

$$t_1 = \lambda x_1, \qquad t_2 = \lambda^{\gamma} x_2$$

  where $\lambda \to 0$ and $\gamma > 0$ is fixed
- $\gamma > 0$ characterizes scaling anisotropy ($\gamma = 1$: isotropic scaling)

$\Gamma := \mathrm{diag}(1,\gamma)$, $\lambda^\Gamma = \mathrm{diag}(1,\lambda^\gamma)$, $\lambda^\Gamma \boldsymbol{t} = (\lambda t_1, \lambda^\gamma t_2) \in \mathbb{R}^2$

Definition

Suppose there exist normalization $d_{\lambda,\gamma} \downarrow 0\, (\lambda \downarrow 0)$ s.t.

$$d_{\lambda,\gamma}^{-1}(X(\boldsymbol{t}_0 + \lambda^\Gamma \boldsymbol{t}) - X(\boldsymbol{t}_0)) \overset{\mathrm{fdd}}{\to} T_\gamma(\boldsymbol{t}), \tag{1}$$

$$d_{\lambda,\gamma}^{-1} X]\boldsymbol{t}_0, \boldsymbol{t}_0 + \lambda^\Gamma \boldsymbol{t}] \overset{\mathrm{fdd}}{\to} V_\gamma(\boldsymbol{t}), \tag{2}$$

T_γ and V_γ in (1), (2) are called γ-tangent and γ-rectangent RFs of RF X at $\boldsymbol{t}_0$ respectively.

- 1-tangent or tangent (isotropic scaling) RF T_1 in (1) was introduced in Falconer (2002) ($\overset{\mathrm{fdd}}{\to}$ replaced by a functional convergence)
- generalizes the concept of *tangent process* for $X = \{X(t)\}$ with $t \in \mathbb{R}$
- normalization $d_{\lambda,\gamma} \downarrow 0$ generally different for (1) and (2)
- dependence on $\boldsymbol{t}_0$ on r.h.s. of (1) and (2) is suppressed (do not depend on $\boldsymbol{t}_0$ by stationarity of increments in this talk)
- 'rectangent' = abridge for 'rectangular tangent'

1. γ-tangent and γ-rectangent RFs

- Since T_1 is self-similar (SS) the existence of T_1 in (1) also termed 'local asymptotic self-similarity' (Benassi, Cohen, Istas (2004), Cohen (2012), Cohen, Istas (2013))
- The above papers proved the existence of T_1 for a class of isotropic fractional Lévy RFs on $\mathbb{R}^d$
- Under mild conditions all scaling limits in (1)–(2) satisfy the (H, γ)-*SS* property:

$$U(\lambda^{\Gamma}\boldsymbol{t}) \stackrel{\mathrm{fdd}}{=} \lambda^{H} U(\boldsymbol{t}), \qquad \forall \lambda > 0, \tag{3}$$

with some $H = H(\gamma) > 0$, normalization $d_{\lambda,\gamma}$ is H-regularly varying as $\lambda \downarrow 0$

Definition

Suppose γ-rectangent limits in (2) exist for any $\gamma > 0$. We say that these limits exhibit scaling transition at some $\gamma_0 > 0$ if

$$V_\gamma = \begin{cases} V_+, & \gamma > \gamma_0, \\ V_-, & \gamma < \gamma_0, \\ V_0, & \gamma = \gamma_0 \end{cases} \qquad \text{and} \qquad V_+ \stackrel{\mathrm{fdd}}{\neq} aV_- \ (\forall a > 0) \tag{4}$$

1. γ-tangent and γ-rectangent RFs

- Analogous definition in γ-tangent case
- Closely related to *large-scale* scaling transition for RFs on $\mathbb{Z}^2$ or $\mathbb{R}^2$:
  Puplinskaitė, Surgailis (2015). Stoch. Proc. Appl. 125, 2256–2271;
  Puplinskaitė, Surgailis (2016). Bernoulli 22, 2401–2441;
  Pilipauskaitė, Surgailis (2016). J. Appl. Prob. 53, 857–879;
  Pilipauskaitė, Surgailis (2017). Stoch. Proc. Appl. 127, 2751–2779;
  Surgailis (2020). Stoch. Proc. Appl. 130, 7518–7546;
  Pilipauskaitė, Surgailis (2021). An Out of Equilibrium 3: Celebrating Vladas Sidoravicius, pp. 683–710;
  Damarackas, Paulauskas (2021). J. Math. Anal. Appl. 497
- In the large-scale case, rectangular increments are replaced by integrals or sums of a stationary RF X over large rectangle $[0, \lambda x_1] \times [0, \lambda^\gamma x_2]$ and one is interested in the limit

$$d_{\lambda,\gamma}^{-1} \int_{[0,\lambda x_1]\times[0,\lambda^\gamma x_2]} X(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \overset{\mathrm{fdd}}{\to} V_\gamma(\boldsymbol{x}), \qquad \lambda \to \infty \tag{5}$$

  for any given $\gamma > 0$
- In the above works a similar trichotomy to (2) was observed in large-scale anisotropic scaling for several classes of linear and nonlinear long-range dependent (LRD) RF models The trichotomy was called the *scaling transition*, with $V_\pm$ the *unbalanced* and V_0 the *well-balanced* scaling limits.
- Intrinsically related to LRD

1. γ-tangent and γ-rectangent RFs

- Large-scale trichitomy or scaling transition of *different nature* occurs in applied sciences (telecommunications and econometrics) in joint temporal-spatial aggregation of *independent* LRD processes:

$$d_{\lambda,\gamma}^{-1} \sum_{i=1}^{[\lambda^\gamma x_2]} \int_0^{\lambda x_1} (X_i(t) - \mathrm{E} X_i(t))\, \mathrm{d}t \ \stackrel{\mathrm{fdd}}{\to} \ V_\gamma(\boldsymbol{x}), \qquad \lambda \to \infty \tag{6}$$

  where $X_i = \{X_i(t), t \in \mathbb{R}\}$ are independent copies of a stationary process $X = \{X(t), t \in \mathbb{R}\}$. Typical examples of X:

  - ON/OFF process with heavy tailed ON or OFF intervals (telecommunications)
  - random-coefficient AR(1) process with random coefficient having a power-law distribution near the unit root (econometrics)

- Gaussian or stable limits in (6) depending on whether $\gamma > \gamma_0$ or $\gamma < \gamma_0$

Mikosch, Resnick, Rootzén, Stegeman (2002). Ann. Appl. Probab. 12, 23–68;
Gaigalas, Kaj (2003). Bernoulli 9, 671–703;
Pipiras, Taqqu, Levy (2004). Bernoulli 10, 121–163;
Kaj, Taqqu (2008). An Out of Equilibrium 2, pp. 383–427;
Pipiras, Taqqu (2017) *Long-Range Dependence and Self-Similarity*. Cambridge Univ. Press, Cambridge;
Pilipauskaitė, Surgailis (2014). Stoch. Proc. Appl. 124, 1011–1035:
Pilipauskaitė, Skorniakov, Surgailis (2020) Adv. Appl. Probab. 52, 237–265

2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

We consider RF $X = \{X(\boldsymbol{t}), \boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2_+\}$ on $\mathbb{R}^2_+ = (0,\infty)^2$ written as stochastic integral:

$$X(\boldsymbol{t}) := \int_{\mathbb{R}^2} \left\{ g(\boldsymbol{t}-\boldsymbol{u}) - g_1^0((t_1,0)-\boldsymbol{u}) - g_2^0((0,t_2)-\boldsymbol{u}) + g_{12}^0(-\boldsymbol{u}) \right\} M(\mathrm{d}\boldsymbol{u}), \qquad (7)$$

w.r.t. Lévy random measure $M(A), A \subset \mathbb{R}^2_+$, with independent values on disjoint sets and characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta M(A)} = \exp\left\{ \mathrm{Leb}(A)\Big(-\frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta \ell_\alpha(y))\nu(\mathrm{d}y)\Big)\right\}, \qquad \theta \in \mathbb{R}, \quad (8)$$

where:

- $\sigma^2 \ge 0$, $0 < \alpha \le 2$ and ν is a Lévy measure on $\mathbb{R}$ satisfying Assumption $(\mathrm{M})_\alpha$ below (meaning roughly that $\lambda^{-2/\alpha} M(\lambda A)$ tends to α-stable random measure $W_\alpha(A)$ as $\lambda \to 0$)
- g, g_1^0, g_2^0, g_{12}^0 are deterministic functions satisfying a power-law behavior at the origin $\mathbf{0} \in \mathbb{R}^2$ (specified in Assumption $(\mathrm{G})_\alpha$)
- centering $\ell_\alpha(y) := y$ $(1 < \alpha \le 2), := 0$ $(0 < \alpha < 1)$

2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

- g_1^0, g_2^0, g_{12}^0 'initial functions' do not enter rectangular increment:
$$X]\mathbf{0},\boldsymbol{t}] = \int_{\mathbb{R}^2} g]-\boldsymbol{u},\boldsymbol{t}-\boldsymbol{u}]M(\mathrm{d}\boldsymbol{u}), \qquad \text{where}$$
$$g]-\boldsymbol{u},\boldsymbol{t}-\boldsymbol{u}] = g(t_1-u_1,t_2-u_2)-g(-u_1,t_2-u_2)-g(t_1-u_1,-u_2)+g(-u_1,-u_2)$$
- X in (7): stationary rectangular increments: $X]\boldsymbol{t}_0,\boldsymbol{t}_0+\boldsymbol{t}] \overset{\mathrm{fdd}}{=} X]\mathbf{0},\boldsymbol{t}]$
- X in (7): direct 2-dim analog of *Lévy driven moving average* with 1-dim time:
$$X(t) = \int_{-\infty}^{t} (g(t-u)-g^0(-u))M(\mathrm{d}u), \qquad t \ge 0$$
with kernel $g(t) \sim ct^q, t \downarrow 0$ [Basse-O'Connor, Lachièze-Rey, Podolskij (2017) *Ann. Probab.* **45**, 4477–4528]
- [Cohen, Istas (2013) *Fractional Fields and Applications.* Mathématiques et Applications **73**, Springer]: 1-tangent limits of *isotropic* fractional Lévy RFs on $\mathbb{R}^d$:
$$X(\boldsymbol{t}) = \int_{\mathbb{R}^d} \{\|\boldsymbol{t}-\boldsymbol{u}\|^{H-\frac{2}{\alpha}} - \|\boldsymbol{u}\|^{H-\frac{2}{\alpha}}\}M(\mathrm{d}\boldsymbol{u})$$
(particular parametric case of (7))
- we consider *anisotropic* power behavior of $g(\boldsymbol{t})$ at $\boldsymbol{t}=\mathbf{0}\in\mathbb{R}^2$ characterized by *two* exponents $q_1,q_2>0$

2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

$$g(\boldsymbol{t}) \sim g_0(\boldsymbol{t}) := \rho(\boldsymbol{t})^{\chi} L(\boldsymbol{t}), \quad \boldsymbol{t} \to \boldsymbol{0}, \tag{9}$$

where:

- $\rho(\boldsymbol{t}) := |t_1|^{q_1} + |t_2|^{q_2}$ (anisotropic radial generalized invariant function)
- $q_1 > 0$, $q_2 > 0$, $\chi \neq 0$: parameters, $Q := \frac{1}{q_1} + \frac{1}{q_2}$
- $L(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^2_0$ (angular generalized invariant function); some regularity conditions
- $\chi > 0$ ($\chi < 0$): (9) vanishes (explodes) at $\boldsymbol{t} = \boldsymbol{0}$
- rectangent limits of RF X depend on *two* parameters only:

$$p_i := q_i(Q - \chi) > 0, \qquad i = 1, 2, \tag{10}$$

Example (Fractional Lévy RF)

$$X(\boldsymbol{t}) = \int_{\mathbb{R}^2} \{\|\boldsymbol{t} - \boldsymbol{u}\|^{H - \frac{2}{\alpha}} - \|\boldsymbol{u}\|^{H - \frac{2}{\alpha}}\} M(\mathrm{d}\boldsymbol{u})$$

2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

- If $EM(\mathrm{d}\boldsymbol{u})^2 = \sigma^2 \mathrm{d}\boldsymbol{u},\ \alpha = 2,\ 0 < H < 1$ then

$$\mathrm{E}X(\boldsymbol{t})X(\boldsymbol{s}) = \mathrm{E}|X(\boldsymbol{e}_1)|^2 \frac{1}{2}(\|\boldsymbol{t}\|^{2H} + \|\boldsymbol{s}\|^{2H} - \|\boldsymbol{t} - \boldsymbol{s}\|^{2H}), \qquad \boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}^2, \qquad (11)$$

- If M Gaussian then X Gaussian (called fractional Brownian RF) (review paper [Lodhia, Scheffield, Sun, Watson, (2016), *Probab. Surv.* **13**, 1–56]) 1-tangent limits [Benassi, A., Cohen, S. and Istas, J. (2004). *Bernoulli* **10**, 357–373], [Cohen, Istas (2013)]
- Satisfies fractional PDE with Laplace operator (particular case of Ex 2)
- Satisfies (9) with $g_0 = g,\ q_1 = q_2 = 2,\ Q = 1,\ \chi = \frac{H}{2} - \frac{1}{\alpha} \in (-\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{\alpha})$.

Example (isotropic Matérn RF)

$$X(\boldsymbol{t}) = \begin{cases} Y(\boldsymbol{t}) - Y(\boldsymbol{0}), & \chi > 0, \\ Y(\boldsymbol{t}), & \chi < 0, \end{cases} \qquad \text{where} \qquad (12)$$

$$(c^2 - \Delta)^{1+\chi} Y(\boldsymbol{t}) = \dot{M}(\boldsymbol{t}) \quad \text{(fractional PDE, } \Delta = \text{Laplace)} \qquad (13)$$

2. Lévy driven fractional RFs on $\mathbb{R}^2$. Examples

- If $\mathrm{E}M(\mathrm{d}\boldsymbol{u})^2 = \sigma^2 \mathrm{d}\boldsymbol{u}$ and $\chi > -\frac{1}{2}$, then $\mathrm{E}|Y(\mathbf{0})|^2 < \infty$ and

$$\mathrm{E}Y(\mathbf{0})Y(\boldsymbol{t}) = \mathrm{E}|Y(\mathbf{0})|^2 \frac{(c\|\boldsymbol{t}\|)^{1+2\chi} K_{1+2\chi}(c\|\boldsymbol{t}\|)}{\Gamma(1+2\chi)2^{2\chi}} \qquad (K_\nu : \text{modif. Bessel function})$$

- Matérn RFs and covariance functions widely used in spatial applications [Guttorp, Gneiting (2006). *Biometrika* **93**, 989–995]
- Satisfies (9) with $g_0(\boldsymbol{t}) = \|\boldsymbol{t}\|^{2\chi}$, $q_1 = q_2 = 2$, $Q = 1$, $\chi \in (-\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{\alpha})$

Example (anisotropic heat operator RF)

$$(c_1 + \Delta_{12})^{\chi + \frac{3}{2}} X(\boldsymbol{t}) = \dot{M}(\boldsymbol{t}), \qquad \boldsymbol{t} \in \mathbb{R}^2, \tag{14}$$

$$\Delta_{12} := \frac{\partial}{\partial t_1} - c_2^2 \frac{\partial^2}{\partial t_2^2} \qquad \text{(heat operator)}$$

- Fundamental solution of (14) (seems new?):

$$g(\boldsymbol{t}) = \frac{t_1^{\chi}}{2^{\frac{1}{2}}(2\pi)^{\frac{3}{2}} c_2 \Gamma(\chi + \frac{3}{2})} \exp\left\{ -c_1 t_1 - \frac{t_2^2}{4c_2^2 t_1} \right\}, \qquad t_1 > 0 \tag{15}$$

- Solution of (14) via Fourier transform: [Kelbert, Leonenko, Ruiz-Medina (2005), *Adv. Appl. Probab.* **37**, 108–133]
- Satisfies (9) with $g_0(\boldsymbol{t}) := \rho(\boldsymbol{t})^{\chi}\ell(\boldsymbol{t}), \rho(\boldsymbol{t}) := |t_1| + |t_2|^2, q_1 := 1,\ q_2 := 2,\ Q = \frac{3}{2}$, and continuous angular function

$$\ell(\boldsymbol{t}) := \frac{z^{\chi}}{2^{\frac{1}{2}}(2\pi)^{\frac{3}{2}} c_2 \Gamma(\chi + \frac{3}{2})} \exp\left\{ - \frac{1}{4c_2^2}\left(\frac{1}{z} - 1\right)\right\},\ t_1 > 0,\ z := \frac{t_1}{\rho(\boldsymbol{t})} \in (0, 1].$$

3. Main results [γ-rectangent limits of Lévy driven RFs]

Definition ([Genton, Perrin, Taqqu (2007). *Stoch. Models* **23**, 397–411])

A RF $V = \{V(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{R}_+^2\}$ *is said* (H_1, H_2)*-multi-self-similar (MSS) with parameters* $H_i \ge 0,\ i = 1, 2$ *if*

$$V(\lambda_1 t_1, \lambda_2 t_2) \overset{\text{fdd}}{=} \lambda_1^{H_1}\lambda_2^{H_2} V(\boldsymbol{t}), \qquad \forall\lambda_1 > 0,\ \forall\lambda_2 > 0. \tag{16}$$

3. Main results

Classical example of MSS RF: *Fractional Brownian Sheet (FBS)* B_{H_1,H_2}: Gaussian process on $\mathbb{R}^2_+$ with zero mean and covariance

$$\mathrm{E}B_{H_1,H_2}(\boldsymbol{t})B_{H_1,H_2}(\boldsymbol{s}) = (1/4)\prod_{i=1}^{2}(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}), \qquad (17)$$

$\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2_+$, $\boldsymbol{s} = (s_1, s_2) \in \mathbb{R}^2_+$

- Usually B_{H_1,H_2} is defined for $H_i \in (0,1]$ or $H_i \in (0,1)$
- We extend B_{H_1,H_2} to $H_i \in [0,1]$ by continuity in (17)
- Extension to $H_1 \wedge H_2 = 0$ leads to very unusual and extremely singular RF (non-measurable paths!)
- Similarly by continuity define FBM B_0 with $H = 0$ as Gaussian process on $\mathbb{R}_+$ with zero mean and covariance $\mathrm{E}B_0(t)B_0(s) = 1 - \frac{1}{2}I(t \neq s)$
- B_0 is self-similar with $H = 0$ and can be represented as $B_0 \overset{\text{fdd}}{=} \{\frac{1}{\sqrt{2}}(W(t) - W(0)),\ t \in \mathbb{R}_+\}$, where $W(t)$, $t \in [0,\infty)$, is (uncountable) family of *independent* $N(0,1)$ r.v.s.

3. Main results

Summary of main results: For Lévy driven RFs in (7) with kernel g satisfying (9) $(g(\boldsymbol{t}) \sim \rho(\boldsymbol{t})^{\chi} L(\boldsymbol{t}),\ \boldsymbol{t} \to \boldsymbol{0}, \rho(\boldsymbol{t}) = |t_1|^{q_1} + |t_2|^{q_2})$ and random measure $M \sim W_\alpha, 0 < \alpha \le 2$

(i) γ-rectangent limits V_γ exist for any $\gamma > 0$ and are α-stable RFs,

(ii) limit family $\{V_\gamma, \gamma > 0\}$ exhibits scaling transition at $\gamma_0 := \frac{q_1}{q_2}$,

(iii) unbalanced limits $V_\pm$ are (H_1, H_2)-MSS with *one of* H_i, $i = 1, 2$ *equal* 1 *or* 0.

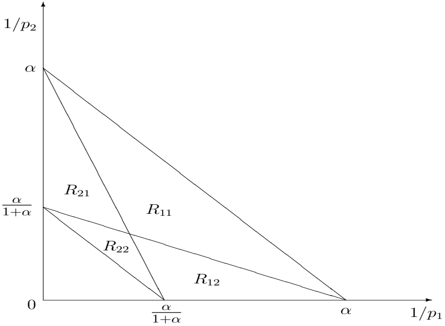
γ-tangent limits (ordinary increments): more straightforward

Conclusions:

- Gaussian unbalanced limits $V_\pm$ agree with B_{H_1,H_2} with $H_1 \wedge H_2 = 0$ or $H_1 \vee H_2 = 1$
- Unbalanced scaling $(\gamma \ne \gamma_0)$ degenerates dependence in one direction (either vertical or horizontal)
- Critical γ_0 agrees with 'intrinsic local dependence ratio' $\frac{q_1}{q_2}$ of RF X

Q. How $H_i, i = 1, 2$ depend on q_1, q_2, χ, α? Parameters $p_1, p_2, \alpha,\ p_i = q_i(Q - \chi) > 0, i = 1, 2, Q = \frac{1}{q_1} + \frac{1}{q_2}$

3. Main results



Parameter region	V_+	Hurst parameters	V_-	Hurst parameters
R_{11}	$\tilde{\Upsilon}_{\alpha,1}$	$0 < H_1 < 1, H_2 = 1$	$\tilde{\Upsilon}_{\alpha,2}$	$H_1 = 1, 0 < H_2 < 1$
R_{12}	$\tilde{\Upsilon}_{\alpha,1}$	$0 < H_1 < 1, H_2 = 1$	$\Upsilon_{\alpha,1}$	$0 < H_1 < 1, H_2 = 0$
R_{21}	$\Upsilon_{\alpha,2}$	$H_1 = 0, 0 < H_2 < 1$	$\tilde{\Upsilon}_{\alpha,2}$	$H_1 = 1, 0 < H_2 < 1$
R_{22}	$\Upsilon_{\alpha,2}$	$H_1 = 0, 0 < H_2 < 1$	$\Upsilon_{\alpha,1}$	$0 < H_1 < 1, H_2 = 0$

1 lentelė: Unbalanced rectangent scaling limits $V_\pm$ and their Hurst parameters in regions R_{ij}, $i, j = 1, 2$.

3. Main results

- Four lines in Fig correspond to $P_{1,1} = \alpha,\ P_{1,1} = \frac{\alpha}{1+\alpha},\ P_{1/\alpha,(1+\alpha)/\alpha} = 1,\ P_{(1+\alpha)/\alpha,1/\alpha} = 1$ where

$$P_{c_1,c_2} := \frac{c_1}{p_1} + \frac{c_2}{p_2}$$

- α-stable RFs $\Upsilon_{\alpha,i}, i = 1,2$ with $H_1 \wedge H_2 = 0$ and $\tilde{\Upsilon}_{\alpha,i}, i = 1,2$ with $H_1 \vee H_2 = 1$ defined through self-similar α-stable processes $Y_{\alpha,i}, \tilde{Y}_{\alpha,i}, i = 1,2$ with 1-dim time, particularly,

$$\tilde{\Upsilon}_{\alpha,1}(\boldsymbol{t}) := t_2 \tilde{Y}_{\alpha,1}(t_1), \qquad \tilde{\Upsilon}_{\alpha,2}(\boldsymbol{t}) := t_1 \tilde{Y}_{\alpha,2}(t_2).$$

- Definition of $\Upsilon_{\alpha,i}$ through $Y_{\alpha,i}$ more involved (only FDD as probability measure on $\mathbb{R}^{\mathbb{R}^2_+}$ using Kolmorogov's consistency theorem)
- Hurst indices:

$$H_{\alpha,1} := \frac{1+\alpha}{\alpha}\left(1 + \frac{p_1}{p_2}\right) - p_1, \quad \tilde{H}_{\alpha,1} := \frac{1+\alpha}{\alpha} + \frac{p_1}{\alpha p_2} - p_1,$$
$$H_{\alpha,2} := \frac{1+\alpha}{\alpha}\left(1 + \frac{p_2}{p_1}\right) - p_2, \quad \tilde{H}_{\alpha,2} := \frac{1+\alpha}{\alpha} + \frac{p_2}{\alpha p_1} - p_2.$$

3. Main results (rigorous formulations)

$\partial_i f(\boldsymbol{t}) := \partial f(\boldsymbol{t})/\partial t_i,\ i = 1,2,\ \partial_{12} f(\boldsymbol{t}) := \partial^2 f(\boldsymbol{t})/\partial t_1 \partial t_2$

$f : \mathbb{R}^2_0 \to \mathbb{R}$ is *generalized homogeneous* (resp., *generalized invariant*) if $\exists\ q_i > 0, i = 1,2$ s.t. $\lambda f(\lambda^{1/q_1} t_1, \lambda^{1/q_2} t_2) = f(\boldsymbol{t})\ \forall\ \lambda > 0,\ \forall\ \boldsymbol{t} \in \mathbb{R}^2_0$ (resp., $f(\lambda^{1/q_1} t_1, \lambda^{1/q_2} t_2)$ does not depend on $\lambda > 0\ \forall\ \boldsymbol{t} \in \mathbb{R}^2_0$).

Gen. homog. function $f(\boldsymbol{t})$ can be represented as $f(\boldsymbol{t}) = \rho(\boldsymbol{t})^{-1} \ell(\boldsymbol{t})$ with $\rho(\boldsymbol{t}) = |t_1|^{q_1} + |t_2|^{q_2}$ and a gen. inv. $\ell(\boldsymbol{t}) = \tilde{\ell}(t_1/\rho(\boldsymbol{t})^{1/q_1}, t_2/\rho(\boldsymbol{t})^{1/q_2})$ where $\tilde{\ell}$ is restriction of f to $\{\boldsymbol{t} \in \mathbb{R}^2_0 : \rho(\boldsymbol{t}) = 1\}$

Assumptions on kernels g, g^0_i, g^0_{12} of Lévy driven RF X.

***Assumption* (G)$_\alpha$.**

- $g_0(\boldsymbol{t}) = \rho(\boldsymbol{t})^\chi L(\boldsymbol{t})$, where $L(\boldsymbol{t})$ is a gen. inv. and $\chi \in \mathbb{R}_0$, $q_i > 0$, $i = 1,2$, with $Q = \frac{1}{q_1} + \frac{1}{q_2}$ s.t.
$$-\frac{1}{\alpha} Q < \chi < \left(1 - \frac{1}{\alpha}\right) Q$$
- As $|\boldsymbol{t}| \to 0$, $g(\boldsymbol{t}) = g_0(\boldsymbol{t}) + o(\rho(\boldsymbol{t})^\chi)$, $\partial_i g(\boldsymbol{t}) = \partial_i g_0(\boldsymbol{t}) + o(\rho(\boldsymbol{t})^{\chi - \frac{1}{q_i}})$, $i = 1,2$, $\partial_{12} g(\boldsymbol{t}) = \partial_{12} g_0(\boldsymbol{t}) + o(\rho(\boldsymbol{t})^{\chi - Q})$ and $\forall\ \boldsymbol{t} \in \mathbb{R}^2_0$,
$$|g_0(\boldsymbol{t})| \le C\rho(\boldsymbol{t})^\chi, \quad |\partial_i g_0(\boldsymbol{t})| \le C\rho(\boldsymbol{t})^{\chi - \frac{1}{q_i}},\ i = 1,2, \quad |\partial_{12} g_0(\boldsymbol{t})| \le C\rho(\boldsymbol{t})^{\chi - Q}.$$

***Assumption* (G)$^0_\alpha$.** For any $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2$, $\delta > 0$,

$$\int_{\mathbb{R}^2} |g(\boldsymbol{t} - \boldsymbol{u}) - g^0_1((t_1, 0) - \boldsymbol{u}) - g^0_2((0, t_2) - \boldsymbol{u}) + g^0_{12}(-\boldsymbol{u})|^\alpha \mathrm{d}\boldsymbol{u} \ < \ \infty \quad (0 < \alpha \le 2),$$
$$\int_{|\boldsymbol{u}|>\delta} \Big(\sum_{i=1}^2 |\partial_i g(\boldsymbol{u})|^\alpha + |\partial_{12} g(\boldsymbol{u})|^\alpha \Big) \mathrm{d}\boldsymbol{u} \ < \ \infty \quad (1 \le \alpha \le 2). \tag{18}$$

Moreover, if $0 < \alpha < 1$, there exist dominating functions $\bar{g}_i(\boldsymbol{u})$, $\bar{g}_{12}(\boldsymbol{u}), \boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2_+$ monotone decreasing in each $u_i > 0, i = 1, 2$, $|\partial_i g(\boldsymbol{u})| \le \bar{g}_i(|u_1|, |u_2|)$, $|\partial_{12} g(\boldsymbol{u})| \le \bar{g}_{12}(|u_1|, |u_2|), |\boldsymbol{u}| > \delta$, satisfying (18) with $\partial_i g, \partial_{12} g$ replaced by $\bar{g}_i, \bar{g}_{12}$.

Assumptions on Lévy random measure M (characteristics (σ, ν))

***Assumption* (M)$_\alpha$.**

- $\alpha = 2, \sigma > 0$ and $\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) < \infty$, or
- $0 < \alpha < 2, \sigma = 0$ and $\lim_{y \downarrow 0} y^\alpha \nu([y, \infty)) = c_+$, $\lim_{y \downarrow 0} y^\alpha \nu((-\infty, -y]) = c_-$ for some $c_\pm \ge 0$, $c_+ + c_- > 0$, $\sup_{y>0} y^\alpha \nu(\{u \in \mathbb{R} : |u| > y\}) < \infty$. Moreover, if $\alpha = 1$ then $\nu(\mathrm{d}y) = \nu(-\mathrm{d}y), y > 0$ is symmetric.

3. Main results (rigorous formulations)

Assumption $(\mathrm{M})_\alpha$ implies that rescaled Lévy Sheet $M(\boldsymbol{t}) = \int_{]\mathbf{0},\boldsymbol{t}]} M(\mathrm{d}\boldsymbol{u})$ tends to α-stable Sheet $W_\alpha(\boldsymbol{t}) = \int_{]\mathbf{0},\boldsymbol{t}]} W_\alpha(\mathrm{d}\boldsymbol{u})$:

$$(\lambda_1\lambda_2)^{-\frac{1}{\alpha}} M(\lambda_1 t_1, \lambda_2 t_2) \overset{\mathrm{fdd}}{\to} W_\alpha(\boldsymbol{t}), \quad \lambda_i \downarrow 0, \quad i = 1, 2,$$

Theorem

Let Lévy driven fractional RF X in (7) satisfy Assumptions $(G)^0_\alpha$, $(G)_\alpha$ and $(M)_\alpha$; $0 < \alpha \le 2$, $\frac{\alpha}{1+\alpha} < P < \alpha$, $P \neq 1$, $P_{\frac{1}{\alpha},\frac{1+\alpha}{\alpha}} \neq 1$, $P_{\frac{1+\alpha}{\alpha},\frac{1}{\alpha}} \neq 1$. Then the γ-rectangent RF in (2) exists for any $\gamma > 0$, $\boldsymbol{t}_0 \in \mathbb{R}^2_+$ and satisfies the trichotomy

$$V_\gamma = \begin{cases} V_+, & \gamma > \gamma_0, \\ V_-, & \gamma < \gamma_0, \\ V_0, & \gamma = \gamma_0, \end{cases} \tag{19}$$

with $\gamma_0 = \frac{q_1}{q_2} = \frac{p_1}{p_2}$, $V_0(\boldsymbol{t}) := \int_{\mathbb{R}^2} g_0] - \boldsymbol{u}, \boldsymbol{t} - \boldsymbol{u}] W_\alpha(\mathrm{d}\boldsymbol{u})$, , and

$$V_- := \begin{cases} \tilde{\Upsilon}_{\alpha,2}, & P_{\frac{1}{\alpha},\frac{1+\alpha}{\alpha}} > 1, \\ \Upsilon_{\alpha,1}, & P_{\frac{1}{\alpha},\frac{1+\alpha}{\alpha}} < 1, \end{cases} \qquad V_+ := \begin{cases} \tilde{\Upsilon}_{\alpha,1}, & P_{\frac{1+\alpha}{\alpha},\frac{1}{\alpha}} > 1, \\ \Upsilon_{\alpha,2}, & P_{\frac{1+\alpha}{\alpha},\frac{1}{\alpha}} < 1. \end{cases} \tag{20}$$

4. Extensions and comments

1. What are γ-rectangent limits of X when $P = \frac{1}{p_1} + \frac{1}{p_2} > \alpha$ ('smooth' kernel $g(\boldsymbol{t})$)?

   Under 'some' conditions for any $\gamma > 0$

   $$\lambda^{-1-\gamma} X]\mathbf{0}, \lambda^{\Gamma}\boldsymbol{t}] \overset{\mathrm{fdd}}{\to} t_1 t_2 V, \quad \text{where} \quad V := \int_{\mathbb{R}^2} \partial_{12} g(\boldsymbol{u}) M(\mathrm{d}\boldsymbol{u})$$

   (no scaling transition)

2. When γ-rectangent limits of X agree with α-stable sheet W_α?

   Assumption $(\mathrm{G})_\alpha$ should be replaced by assuming that $g(\boldsymbol{t})$ is discontinuous at $\boldsymbol{t} = \mathbf{0}$ and exist 'limites quadrantales' $g_{ij} := \lim_{|\boldsymbol{t}|\to 0,\, \boldsymbol{t}\in\mathbb{R}^2_{ij}} g(\boldsymbol{t})$, $i, j \in \{1, -1\}$ on each quadrant $\mathbb{R}^2_{ij} := \{\boldsymbol{t} \in \mathbb{R}^2 : \mathrm{sgn}(t_1) = i, \mathrm{sgn}(t_2) = j\}$, $i, j = \pm 1$ with

   $$g[\mathbf{0}] := \sum_{i,j\in\{1,-1\}} ij\, g_{ij} \neq 0.$$

   (no scaling transition)

3. What are γ-tangent limits (ordinary increments) of X?

   For RF $X(\boldsymbol{t}) = \int_{\mathbb{R}^2} \left\{ g(\boldsymbol{t} - \boldsymbol{u}) - g^0_{12}(-\boldsymbol{u}) \right\} M(\mathrm{d}\boldsymbol{u})$ with stationary increments and some related conditions on $g(\boldsymbol{t})$ and M we prove that γ-tangent limits T_γ in (1) of X exist for any $\gamma > 0$ and

4. Extensions and comments

$$T_\gamma = \begin{cases} T_+, & \gamma > \gamma_0, \\ T_-, & \gamma < \gamma_0, \\ T_0, & \gamma = \gamma_0, \end{cases}$$

where $\gamma_0 = \frac{q_1}{q_2}$, $T_0(\boldsymbol{t}) := \int_{\mathbb{R}^2} \big\{ g_0(\boldsymbol{t}-\boldsymbol{u}) - g_0(-\boldsymbol{u}) \big\} W_\alpha(\mathrm{d}\boldsymbol{u})$ and $T_+(\boldsymbol{t}) := T_0(t_1, 0)$, $T_-(\boldsymbol{t}) := T_0(0, t_2)$ *depend on only one coordinate* on the plane.

4. Extension to Lévy driven RFs on $\mathbb{R}^d_+, d \ge 3$: seems possible but open. Description of rectangent limits more complicated. [Surgailis, D. (2019). Anisotropic scaling limits of long-range dependent linear random fields on $\mathbb{Z}^3$. *J. Math. Anal. Appl.* **472**, 328–351], [Damarackas,J., Paulauskas, V. (2021). *J. Math. Anal. Appl.* **497**].
5. Functional convergence instead of FDD: open. However, if $H_1 \wedge H_2 = 0$ does not seem feasible.
6. Applications to statistical estimation of H_1, H_2 from dense rectangular grid: open and challenging problem.