HYDRODYNAMIC LIMIT OF AN EXCLUSION PROCESS WITH VORTICITY

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STOCHASTIC INTERACTING PARTICLE SYSTEMS EVOLVING ON A LATTICE $\Lambda$ DEPENDING ON A PARAMETER $N$.

- $\eta$: CONFIGURATION OF PARTICLES

- $\eta_\varepsilon(x)$: NUMBER OF PARTICLES AT $x \in \Lambda$ AT TIME $\varepsilon$.

- EXCLUSION RULE: $\eta_\varepsilon(x) = \begin{cases} 1 \\ 0 \end{cases}$ AT MOST 1 PARTICLE ON EACH SITE
\( \Lambda \subseteq \mathbb{R}^d \text{ domain, } \Lambda_N \subseteq \Lambda \); \( \Lambda_N = \text{LATTICE OF SIZE} \ \frac{1}{N} \)

- \( \Lambda = 2\text{-D TORUS} \)
- \( \text{SCALING LIMIT } N \rightarrow +\infty; \ \text{LATTICE MESH IS GOING TO ZERO} \)
**GENERATOR:**

\[ \delta_N f(n) = \sum_{(x,y) \in E_N} \xi_{xy}(n) \left( f(n^{xy}) - f(n) \right) \]

- \( \xi_{xy}(n) = \text{RATE OF JUMP OF ONE PARTICLE FROM } x \text{ TO } y \)
- \( n_{xy}(z) = \begin{cases} n(y) & z = x \\ n(x) & z = y \\ n(z) & z \neq x, y \end{cases} \)
- \( E_N = \text{ORIENTED BONDS OF THE LATTICE} \)
The simplest example is the SEP: symmetric exclusion process for which

\[ C_{xy}(\eta) = \eta(x) \left( 1 - \eta(y) \right) \]

The interaction among the particles is only hard core.

We will consider more general transition rates.
\[ \eta \in \{0, 1\}^\Lambda \rightarrow \Pi_N(\eta) \in \mathcal{P}(\Lambda) \]

\[ \Pi_N(\eta) = \text{EMPIRICAL MEASURE ASSOCIATED TO THE CONFIGURATION } \eta \]

\[ \Pi_N(\eta) = \frac{1}{N^d} \sum_{x \in \Lambda_N} \eta(x) \delta_x \]

\[ \bullet = \text{PARTICLE} \rightarrow \delta \text{ MEASURE WITH WEIGHT } N^{-d} \]
EMPIRICAL CURRENT

- Consider $(m_t)_{t \in [0,1]}$ a trajectory.

- Define $N_{xy}(t) = \# \text{jumps from } x \to y$ in $(0,t]$

- Define $J_{xy}(t) = \sum_{y \neq x} N_{xy}(t) - N_{yx}(t)$ the current across the bond $(x,y) \in E_N$

- $J_{xy}(t) = -J_{yx}(t)$ antisymmetric; it is a discrete vector field.
Given a smooth vector field on \( \wedge \mathbb{R}^2 \) (recall \( D = 2 \)) —

**DISCRETIZED VERSION** \( G_n \)

\[
G_n(x, y) = \int_{(x, y)} G \cdot dl \quad \text{(LINE INTEGRAL)}
\]

- \( G_n \) is a discrete vector field
- \( G_n(x, y) = O \left( \frac{1}{n} \right) \)
- If \( G = \nabla h \) then \( G_n(x, y) = h(y) - h(x) \)
**EMPIRICAL CURRENT**

\[ (m_t)_{t \in [0, T]} \in D([0, T], \{0, 1\}^N) \implies \]

\[ (J_t^N)_{t \in [0, T]} \in D([0, T], H_{-k}(\Lambda)) \]

**THE EMPIRICAL MEASURE**

\[ \forall t, \forall G \quad \text{WE HAVE} \]

\[ \int_{[0, T]}^N (G) = \frac{1}{2N^d} \sum_{(x,y) \in \mathcal{E} \mathcal{H}} \int_t J_{xy}(t) G_N(x, y) \]
• Given \( \phi, \psi \) two discrete vector fields \( \langle \phi, \psi \rangle \) scalar product defined by

\[
\langle \phi, \psi \rangle = \frac{1}{2} \sum_{(x,y) \in \mathbb{N}} \phi(x,y) \psi(x,y)
\]

• Empirical current

\[
\tilde{J}^N_t(g) = \langle \frac{1}{N} J, g_N \rangle
\]

• Empirical current and empirical measure are related by a continuity equation
- Discrete Continuity Equation

\[ \eta_t^\dagger(x) - \eta_0(x) + (\nabla \cdot J_t)(x) = 0 \]

- \( \nabla \cdot \) = Discrete Divergence

\[ (\nabla \cdot J_t)(x) = \sum_{y : (x, y) \in E_N} J_{xy}(t) \quad y \in \{ x, y \} \in E_N \]

- Summing over \( x \) \( \forall f : \forall \rightarrow \mathbb{R} \) smooth

We have the continuity equation

\[ \int_{\wedge} f d\pi_N^\dagger(\eta_t) - \int_{\wedge} f d\pi_N(\eta_0) = \int_{\wedge} J_t^N (\nabla f) \]
To refresh notation we recall

\[ \int f \, d\pi_N(n) = \frac{1}{N^d} \sum_{x \in \Lambda_N} f(x) \eta(x) \]

and

\[ \int_t \nabla f = \frac{1}{2N^d} \sum_{(x,y) \in \Lambda^2_N} \mathcal{J}_{xy}(t)(f(y) - f(x)) \]
**SCALING LIMIT**

- **Fix initial condition** \( \eta(t) \) **in such a way**
  that\[ \lim_{N \to \infty} N^2 \int \eta(0) \, dx = p_0(x) \, dx \]

- **Under general condition (reversibility, strong mixing, no phase transition)** **we have an**
  hydrodynamic behavior **under a diffusive rescaling**

\[ \lim_{n \to \infty} \mathbb{E} \left( \left| \int \eta_t \, dx - \int p(x(t)) \, dx \right| > \epsilon \right) = 0 \]

\[ \forall f, \forall \epsilon \]
\( p(x,t) \) solves the Cauchy problem

\[
\begin{align*}
\partial_t p &= \nabla \cdot (D(p) \nabla p) \\
p(x,0) &= p_0(x)
\end{align*}
\]

- \( D(p) \) is positive definite symmetric matrix called the diffusion matrix.

- For special models (the gradient ones) \( D \) can be computed. Otherwise \( D(p) \) has a variational characterization (Varadhan-Yau).
The models are conservative and the limiting equation can be written as a conservation law

$$\partial_t p + \nabla \cdot J(p) = 0$$

- $J(p)$ = typical macroscopic current of the system associated to the density $p$.

For reversible gradient models we have

$$J(p) = -D(p) \nabla p$$

(Fick's Law)

(SEP, Zero Range, KMP, Macroscopic Fluctuation Theory...)
We construct a class of models for which Fick's Law is violated (D > 2 for simplicity).

We have

\[ J(r) = -D(r) \nabla \rho - A(r) \nabla \rho \]

\( A(r) \) is an antisymmetric 2x2 matrix and therefore \( \nabla \cdot (A(r) \nabla \rho) = 0 \) and does not contribute to the evolution of the density.
• It is difficult to combine the gradient condition with the knowledge of the invariant measure (usually by reversibility)
• D=2 SEP + very few other known models
• Our models are not reversible, yet diffusive. Have a Bernoulli product invariant measure and satisfy a generalized gradient condition
Fully explicit computations are possible and we have

$$D(p) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A(p) = \begin{pmatrix} 0 & -q'(p) \\ \frac{q'(p)}{q(p)} & 0 \end{pmatrix}$$

With

$$q(p) = 2\alpha [p(1-p)]^2$$

$\alpha$ is a parameter $1|\alpha|<1$. 
A key object to understand hydrodynamic limits is the instantaneous current.

For any configuration $\eta$, this is defined as

$$J^\eta_m(x; \eta) = C_{xy}(\eta) - C_{yx}(\eta)$$

For any $\eta$, $J^\eta_m$ is a discrete vector field.

$$J^\eta_m(x; \eta) = \text{(rate of jump $x \rightarrow y$)} - \text{(rate of jump $y \rightarrow x$)}$$
INSTANTANEOUS CURRENT IS RELEVANT SINCE

\[ M_t(x, y) = J_{xy}(t) - \int_0^t J_{m(s)}(x, y) \, ds \]

IS A MARTINGALE.

GRADIENT CONDITION: A MODEL IS OF GRADIENT TYPE IF THERE EXISTS A LOCAL FUNCTION \( h(n) \) SUCH THAT

\[ J_{n}(x, y) = \xi_y h(n) - \xi_x h(n) \]

\( \xi_2 = \text{TRANSLATION AT } \xi \in \Lambda \)
Example of Gradient Model SEP

\[ J_{\eta}(x, y) = \eta(x)(1-\eta(y)) - \eta(y)(1-\eta(x)) \]

\[ = \eta(x) - \eta(y) = 2y h(n) - 2x h(n) \]

with

\[ h(n) = -\eta(0) \]
By Doob Inequality the scaling behavior of

\[ J^N_t(G) = \frac{1}{2N^d} \sum_{(\eta, \gamma) \in \mathbb{E}_N} J_{\gamma}(\xi) G^N(\xi, \gamma) \]

is equivalent to the one of

\[ \frac{1}{2N^d} \int_0^t \sum_{(\eta, \gamma) \in \mathbb{E}_N} J^N_{s}(\xi, \gamma) G^N(\xi, \gamma) ds \]

If \( J^N_{s}(\xi, \gamma) = 2y h(\eta) - 2x h(\eta) \) it is possible to perform an integration by parts.
We obtain in this way

\[-\frac{1}{\sqrt{d}} \sum_{X \in \Lambda_n} \int_0^t ds \nabla \cdot G_n(x) \]

A key step is now a replacement lemma that implies the above term converges when \( n \to +\infty \) to

\[-\sum_{X \in \Lambda} \int_0^t ds \nabla \cdot H(p(\chi(s))) \nabla \cdot G(x) \]

where

\[ H(p) = E_{V_p}(h) \]

\( V_p = \text{grand canonical invariant measure of density } p. \)
The above result is a weak form of

\[ J(\rho) = H(\rho) \bigtriangledown \rho \]

This implies Fick's law and

\[ D(\rho) = -H'(\rho) \bigtriangledown \rho \]
Discrete Hodge Decomposition

- Any discrete vector field \( \Phi \) can be uniquely orthogonally decomposed as
  \[
  \phi = \phi^\nabla + \phi^\delta + \phi^\Gamma
  \]

  Orthogonal with respect to \( \langle , \rangle \)

  \[
  \phi^\nabla(x,y) = f(y) - f(x) \quad \text{Gradient}
  \]

  \[
  \phi^\delta(x,y) = \Gamma(f^+) - \Gamma(f^-) \quad \text{Circulation}
  \]

  \[
  \Gamma = 2\cdot \text{form}
  \]
\[
\phi_H = c_1 \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) + c_2 \left( \begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\end{array} \right)
\]

HARMONIC
$J_m = J_m^\nabla + J_m^\delta + J_m^H$

$J_m^\nabla (x, y) = \nabla_y h(m) - \nabla_x h(m)$

$J_m^\delta (x, y) = Z_f + g(\eta) - Z_f - g(\eta)$

$J_m^H (x, y) = C_1(\eta) \left( \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} \right) + C_2(\eta) \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \end{array} \right)$
Our class of models is such that $\nabla \cdot \mathbf{J}^H = 0$ and $\mathbf{J}^D$ are such that $g(n)$ and $h(n)$ are local functions.

\[ \forall x \in \mathbb{R}, \forall n, \nabla \cdot \mathbf{J}^D(x) = 0 \]

And this part does not contribute to the evolution of the density.
• We have \( h(n) = -\eta(0) \) as in the SEP.

• \( g(n) \) is a local function that depends just on the occupations of the lattice sites \( \frac{\eta^2}{N} \).

• If \( \eta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \)

  then \( g(n) = 0 \quad \text{if} \quad |\alpha| < 1 \quad \text{otherwise} \)

\[ g(n) = 0 \]
Faces containing exactly 2 particles on opposite vertices are called activated.
The rate of jump of a particle depends on the occupation numbers of the neighboring faces.

The jump rate of the particle $\bullet$ to the empty site $O$ depends on the states $x = \{0\}$. 

```
  o
 / \
E  x
 / \
  o
|
/ \\  
O   x
 / \
= o
```
The values of the rates are the followings:
The model is not reversible, yet has a diffusive behavior.

\( V_0 \) Bernoulli product measures of any parameter \( \alpha \), \( \beta \) are invariant.

Not trivial fact to verify

\[
\sum c_{xy}(n) = \sum c_{yx}(n^{xy})
\]

Reorganize all terms containing \( z_f \) for any given face \( f \) and then check by direct inspection.
MUITO

OBRIGADO!