HYDRODYNAMIC LIMIT OF AN EXCLUSION PROCESS WITH VORTICITY

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- STOCHASTIC INTERACTING PARTICLE SYSTEMS EVOLVING ON A LATTICE Λ_N DEPENDING ON A PARAMETER N.

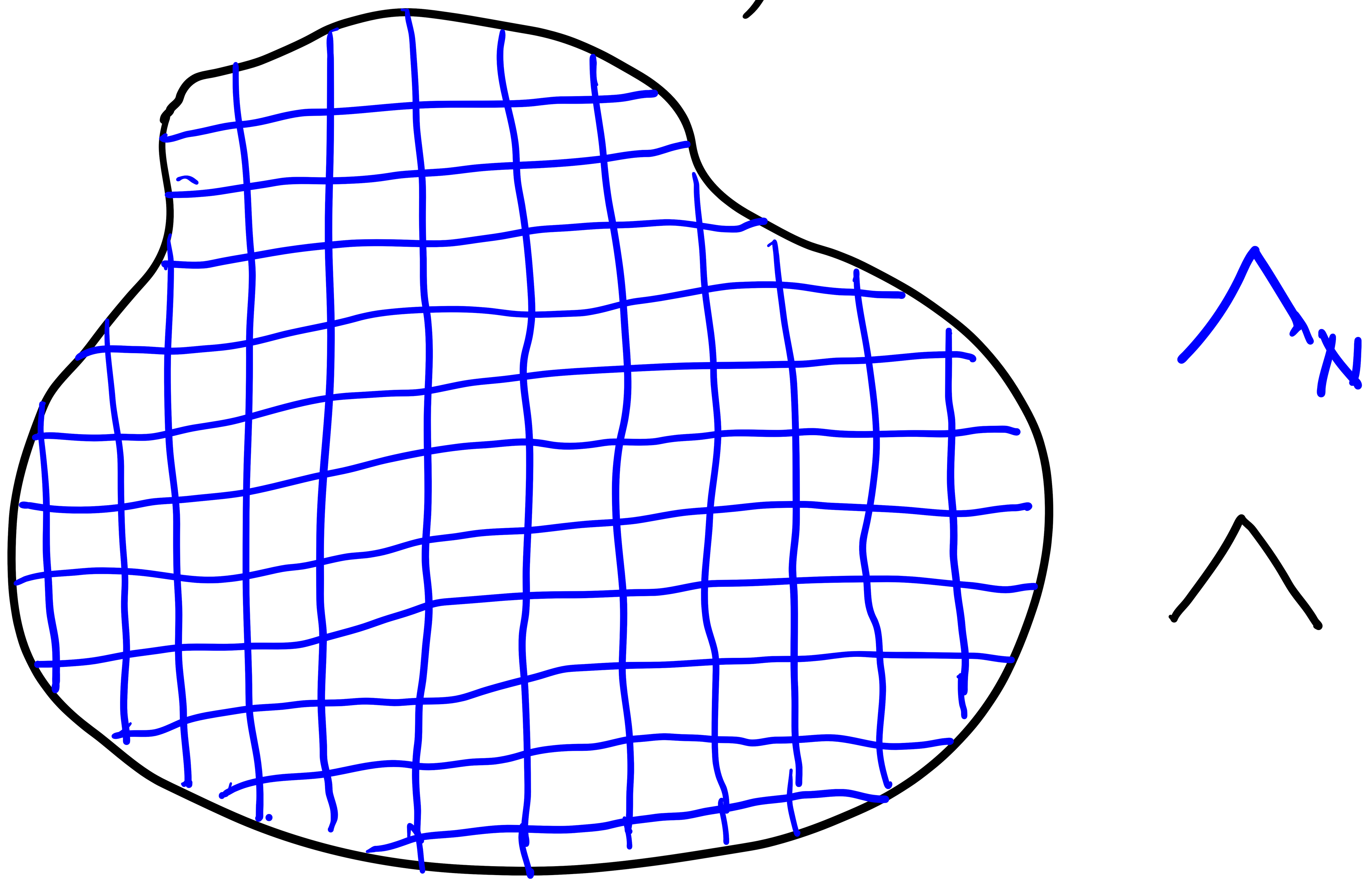
- η = CONFIGURATION OF PARTICLES

- $\eta_t(x)$ = NUMBER OF PARTICLES AT $x \in \Lambda_N$ AT TIME t.

- EXCLUSION RULE $\eta_t(x) = \begin{cases} 1 \\ 0 \end{cases}$

  AT MOST 1 PARTICLE ON EACH SITE

- $\Lambda \subseteq \mathbb{R}^d$ DOMAIN, $\Lambda_N \subseteq \Lambda$; Λ_N = LATTICE OF SIZE $1/N$



- Λ = 2-D TORUS

- SCALING LIMIT $N \to +\infty$; LATTICE MESH IS GOING TO ZERO

- GENERATOR:

$$\mathcal{L}_N f(\eta) = \sum_{(x,y)\in E_N} c_{xy}(\eta)\left(f(\eta^{x,y}) - f(\eta)\right)$$

- $c_{xy}(\eta)$ = RATE OF JUMP OF ONE PARTICLE FROM x TO y
- $$\eta^{xy}(z) = \begin{cases} \eta(y) & z = x \\ \eta(x) & z = y \\ \eta(z) & z \neq x,y \end{cases}$$
- E_N = ORIENTED BONDS OF THE LATTICE

- THE SIMPLEST EXAMPLE IS THE SEP = SYMMETRIC EXCLUSION PROCESS FOR WHICH

$$c_{xy}(\eta) = \eta(x)\big(1-\eta(y)\big)$$

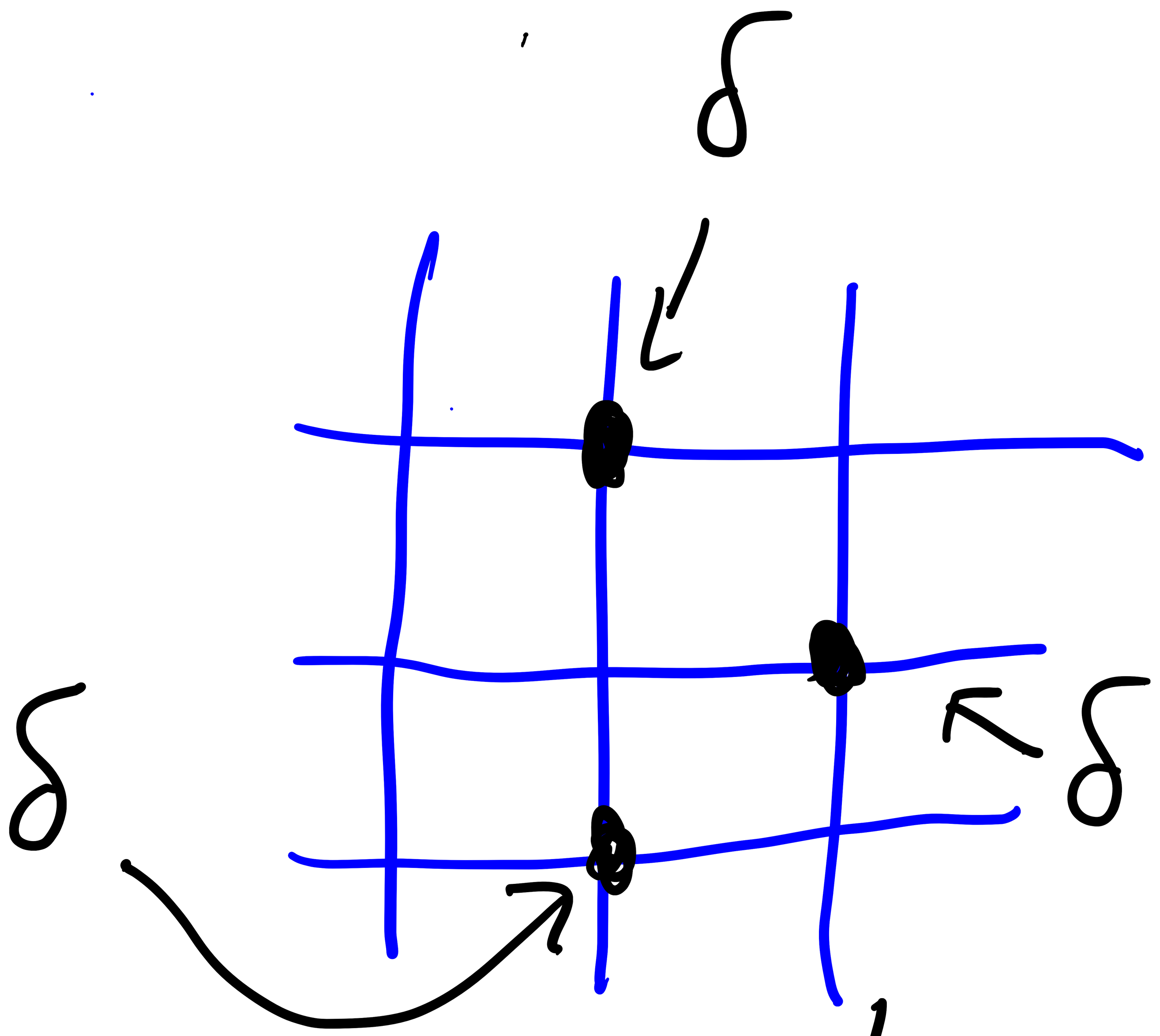
- THE INTERACTION AMONG THE PARTICLES IS ONLY HARD CORE
- WE WILL CONSIDER MORE GENERAL TRANSITION RATES.

EMPIRICAL MEASURE

- $\eta \in \{0,1\}^{\Lambda_N} \longrightarrow \pi_N(\eta) \in \mathcal{M}^+(\Lambda)$

$\pi_N(\eta)$ = EMPIRICAL MEASURE ASSOCIATED TO THE CONFIGURATION η

$$\boxed{\pi_N(\eta) = \frac{1}{N^d} \sum_{x \in \Lambda_N} \eta(x)\, \delta_x}$$



- = PARTICLE $\to$ δ MEASURE WITH WEIGHT N^{-d}

EMPIRICAL CURRENT

- CONSIDER $(\eta_t)_{t\in[0,T]}$ A TRAJECTORY

- DEFINE $N_{xy}(t) = \#$ JUMPS FROM $x \to y$ in $[0,t]$

- DEFINE $J_{xy}(t) = N_{xy}(t) - N_{y,x}(t)$ THE CURRENT ACROSS THE BOND $(x,y) \in E_N$

- $J_{xy}(t) = - J_{y,x}(t)$ ANTISYMMETRIC;

IT IS A DISCRETE VECTOR FIELD

EMPIRICAL CURRENT

- GIVEN G A SMOOTH VECTOR FIELD ON Λ $\quad G : \Lambda \to \mathbb{R}^2$ (RECALL $D=2$) $\longrightarrow$ $\boxed{\text{DISCRETIZED VERSION } G_N}$

$$G_N(x,y) = \int_{(x,y)} G \cdot d\ell \qquad \left(\text{LINE INTEGRAL} \quad (x,y) \in E_N\right)$$

- G_N IS A DISCRETE VECTOR FIELD
- $G_N(x,y) = O(1/N)$
- IF $G = \nabla h$ THEN $G_N(x,y) = h(y) - h(x)$

EMPIRICAL CURRENT

- $(\eta_t)_{t\in[0,T]} \in D\left([0,T], \{0,1\}^{\Lambda_N}\right) \longrightarrow$

$$(\mathcal{J}^N_t)_{t\in[0,T]} \in D\left([0,T], H_{-k}(\Lambda)\right)$$ THE EMPIRICAL MEASURE

- $\forall t, \ \forall G$ WE HAVE

$$\boxed{\mathcal{J}^N_t(G) = \frac{1}{2N^d} \sum_{(x,y)\in E_N} J_{xy}(t)\, G_N(x,y)}$$

- GIVEN ϕ, ψ TWO DISCRETE VECTOR FIELDS $\langle \phi, \psi \rangle$ SCALAR PRODUCT DEFINED BY

$$\langle \phi, \psi \rangle = \frac{1}{2} \sum_{(x,y) \in E_N} \phi(x,y)\, \psi(x,y)$$

- EMPIRICAL CURRENT

$$\mathcal{J}_t^N(G) = \left\langle \frac{1}{N^d} J, G_N \right\rangle$$

- EMPIRICAL CURRENT AND EMPIRICAL MEASURE ARE RELATED BY A CONTINUITY EQUATION

- DISCRETE CONTINUITY EQUATION

$$\eta_t(x) - \eta_0(x) + (\nabla \cdot J_t)(x) = 0$$

- $\nabla \cdot$ = DISCRETE DIVERGENCE

$$(\nabla \cdot J_t)(x) = \sum_{y:(x,y)\in E_N} J_{xy}(t)$$

- SUMMING OVER x $\forall f : \Lambda \to \mathbb{R}$ SMOOTH

WE HAVE THE CONTINUITY EQUATION

$$\boxed{\int_\Lambda f \, d\pi_N(\eta_t) - \int_\Lambda f \, d\pi_N(\eta_0) = \mathcal{J}_t^N(\nabla f)}$$

- TO REFRESH NOTATION WE RECALL

$$\int_{\Lambda} f \, d\pi_N(\eta) = \frac{1}{N^d} \sum_{x \in \Lambda_N} f(x)\, \eta(x)$$

AND

$$\mathcal{J}_t^N(\nabla f) = \frac{1}{2N^d} \sum_{(x,y) \in E^N} J_{xy}(t) \big(f(y) - f(x)\big)$$

SCALING LIMIT

- FIX INITIAL CONDITION $\eta(0)$ IN SUCH A WAY THAT $\lim_{N \to +\infty} \pi_N(\eta(0)) = \rho_0(x)\,dx$

- UNDER GENERAL CONDITION (REVERSIBILITY/ STRONG MIXING, NO PHASE TRANSITION) WE HAVE AN HYDRODYNAMIC BEHAVIOR UNDER A DIFFUSIVE RESCALING $\mathcal{L}_N \to N^2 \mathcal{L}_N$

$$\lim_{N \to +\infty} \mathbb{P}\left(\left| \int f \, d\pi_N(\eta_t) - \int \rho(x,t) f(x)\, dx \right| > \varepsilon \right) = 0$$

$\forall f, \forall \varepsilon$

- $\rho(x,t)$ SOLVES THE CAUCHY PROBLEM

$$\begin{cases} \partial_t \rho = \nabla \cdot \left(D(\rho) \nabla \rho \right) \\ \rho(x,0) = \rho_0(x) \end{cases}$$

- $D(\rho)$ = POSITIVE DEFINITE SYMMETRIC MATRIX CALLED THE <u>DIFFUSION MATRIX</u>
- FOR SPECIAL MODELS (THE GRADIENT ONES) D CAN BE COMPUTED. OTHERWISE $D(\rho)$ HAS A VARIATIONAL CARACHTERIZATION (VARADHAN-YAU)

- THE MODELS ARE CONSERVATIVE AND THE LIMITING EQUATION CAN BE WRITTEN AS A CONSERVATION LAW

$$\partial_t \rho + \nabla \cdot J(\rho) = 0$$

- $J(\rho)$ = TYPICAL MACROSCOPIC CURRENT OF THE SYSTEM ASSOCIATED TO THE DENSITY ρ.

- FOR REVERSIBLE GRADIENT MODELS WE HAVE

$$J(\rho) = -D(\rho)\nabla\rho$$

Fick's Law

(SEP, ZERO RANGE, KMP, MACROSCOPIC FLUCTUATION THEORY.....)

- WE CONSTRUCT A CLASS OF MODELS FOR WHICH FICK'S LAW IS VIOLATED ($D=2$ FOR SIMPLICITY)
- WE HAVE

$$J(\rho) = -D(\rho)\nabla\rho - A(\rho)\nabla\rho$$

- $A(\rho)$ IS AN ANTISYMMETRIC 2×2 MATRIX AND THEREFORE $\nabla\cdot(A(\rho)\nabla\rho) = 0$ AND DOES NOT CONTRIBUTES TO THE EVOLUTION OF THE DENSITY.

- IT IS DIFFICULT TO COMBINE THE GRADIENT CONDITION WITH THE KNOWLEDGE OF THE INVARIANT MEASURE (USUALLY BY REVERSIBILITY)
- D=2 SEP + VERY FEW OTHER KNOWN MODELS
- OUR MODELS ARE NOT REVERSIBLE, YET DIFFUSIVE, HAVE A BERNOULLI PRODUCT INVARIANT MEASURE AN SATISFY A GENERALIZED GRADIENT CONDITION

- FULLY EXPLICT COMPUTATIONS ARE POSSIBLE AND WE HAVE

$$D(\rho) = \mathbb{1} \quad (2\times 2)$$

$$A(\rho) = \begin{pmatrix} 0 & -Q'(\rho) \\ Q'(\rho) & 0 \end{pmatrix}$$

WITH
$$Q(\rho) = 2\alpha \left[\rho(1-\rho)\right]^2$$

α IS A PARAMETER $|\alpha| < 1$.

- A KEY OBJECT TO UNDERSTAND HYDRODYNAMIC LIMITS IS THE <u>INSTANTANEOUS CURRENT</u>

- FOR ANY CONFIGURATION η THIS IS DEFINED AS

$$\boxed{j_\eta(x,y) = c_{xy}(\eta) - c_{yx}(\eta)}$$

- FOR ANY η j_η IS A DISCRETE VECTOR FIELD

- $j_\eta(x,y) = (\text{RATE OF JUMP } x \to y) - (\text{RATE OF JUMP } y \to x)$

- INSTANTANEOUS CURRENT IS RELEVANT SINCE

$$M_t(x,y) = J_{xy}(t) - \int_0^t j_{\eta(s)}(x,y)\, ds$$

IS A MARTINGALE.

- GRADIENT CONDITION: A MODEL IS OF GRADIENT TYPE IF THERE EXISTS A LOCAL FUNCTION $h(\eta)$ SUCH THAT

$$\boxed{j_\eta(x,y) = \tau_y h(\eta) - \tau_x h(\eta)}$$

τ_z = TRANSLATION AT $z \in \Lambda_N$

- EXAMPLE OF GRADIENT MODEL SEP

$$J_\eta(x,y) = \eta(x)\big(1-\eta(y)\big) - \eta(y)\big(1-\eta(x)\big)$$

$$= \eta(x) - \eta(y) = z_y h(\eta) - z_x h(\eta)$$

WITH

$$h(\eta) = -\eta(0)$$

- BY DOOB INEQUALITY THE SCALING BEHAVIOR OF

$$\mathcal{J}_t^N(G) = \frac{1}{2N^d} \sum_{(x,y)\in E_N} J_{xy}(t)\, G_N(x,y)$$

IS EQUIVALENT TO THE ONE OF

$$\frac{1}{2N^d} \int_0^t \sum_{(x,y)\in E_N} J_{\eta(s)}(x,y)\, G_N(x,y)\, ds$$

- IF $J_\eta(x,y) = \tau_y h(\eta) - \tau_x h(\eta)$ IT IS POSSIBLE TO PERFORM AN INTEGRATION BY PARTS

- WE OBTAIN IN THIS WAY

$$-\frac{1}{N^d}\sum_{x\in\Lambda_N}\int_0^t ds\, \tau_x h(\eta(s))\, \nabla\cdot G_N(x)$$

- A KEY STEP IS NOW A REPLACEMENT LEMMA THAT IMPLIES THE ABOVE TERM CONVERGES WHEN $N\to+\infty$ TO

$$-\int_\Lambda dx \int_0^t ds\, H(\rho(x,s))\, \nabla\cdot G(x)$$

WHERE

$$\boxed{H(\rho)=\mathbb{E}_{\nu_\rho}(h)}$$

ν_ρ = GRAND CANONICAL INVARIANT MEASURE OF DENSITY ρ.

- THE ABOVE RESULT IS A WEAK FORM OF

$$J(\rho) = H'(\rho)\,\nabla\rho$$

- THIS IMPLIES FICK'S LAW AND

$$\boxed{D(\rho) = -H'(\rho)\,\mathbb{1}}$$

Discrete Hodge Decomposition

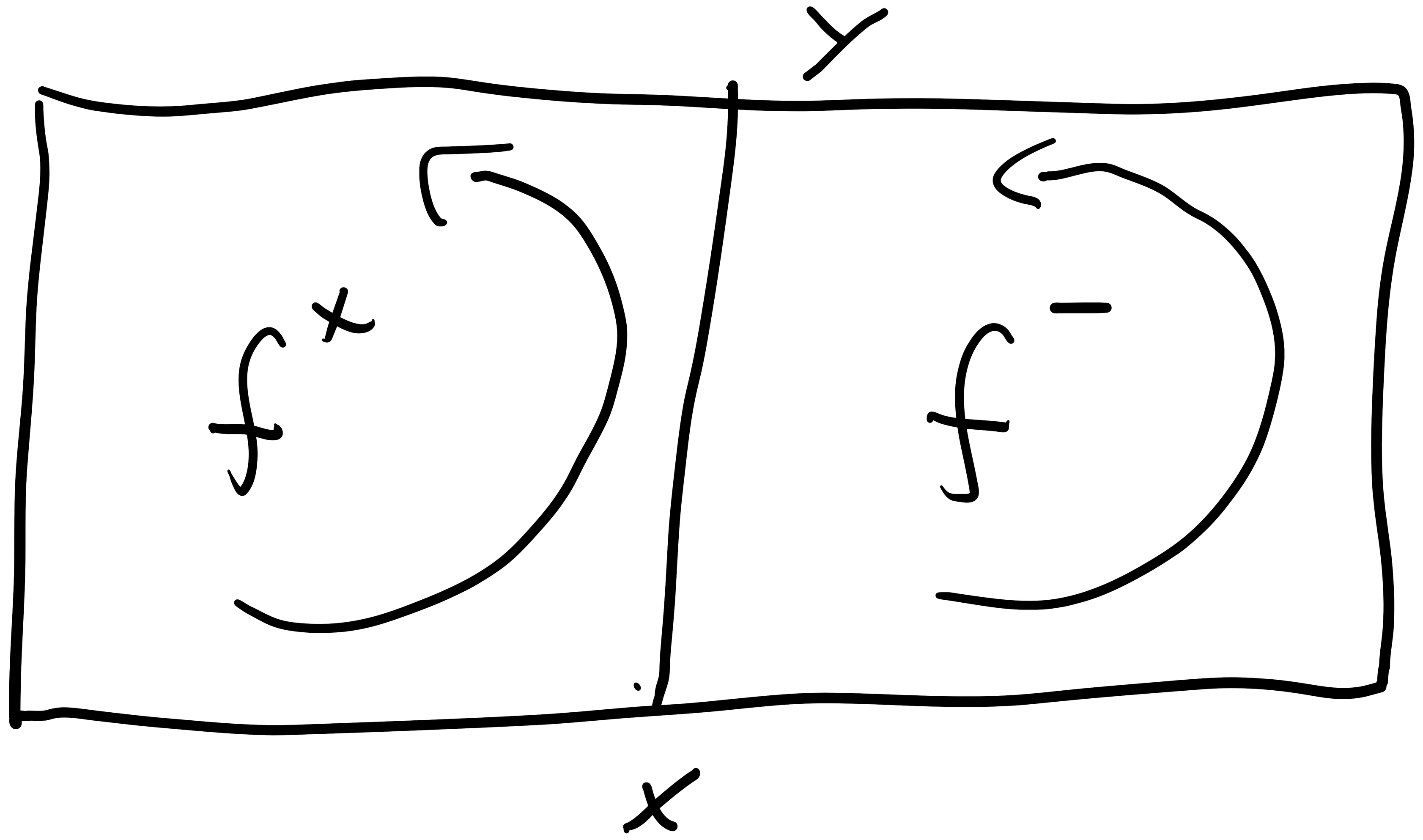
- Any discrete vector field ϕ can be uniquely orthogonally decomposed as

$$\phi = \phi^{\nabla} + \phi^{\delta} + \phi^{H}$$

- Orthogonal with respect to $\langle \, , \, \rangle$
- $\phi^{\nabla}(x,y) = f(y) - f(x)$ GRADIENT
- $\phi^{\delta}(x,y) = \Gamma(f^+) - \Gamma(f^-)$

$\boxed{\Gamma = \text{2-form}}$

CIRCULATION



$$\phi^{H} = c_1 \begin{pmatrix} \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow \end{pmatrix} + c_2 \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow \end{pmatrix}$$

HARMONIC

- J_n TRANSLATIONAL COVARIANT DISCRETE VECTOR FIELD

$$J_n = J_n^{\nabla} + J_n^{\delta} + J_n^{H}$$

- $J_n^{\nabla}(x,y) = \partial_y h(n) - \partial_x h(n)$
- $J_n^{\delta}(x,y) = \partial_{f^+} \rho(n) - \partial_{f^-} \rho(n)$
- $J_n^{H}(x,y) = C_1(n) \begin{pmatrix} \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow \end{pmatrix} + C_2(n) \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow \end{pmatrix}$

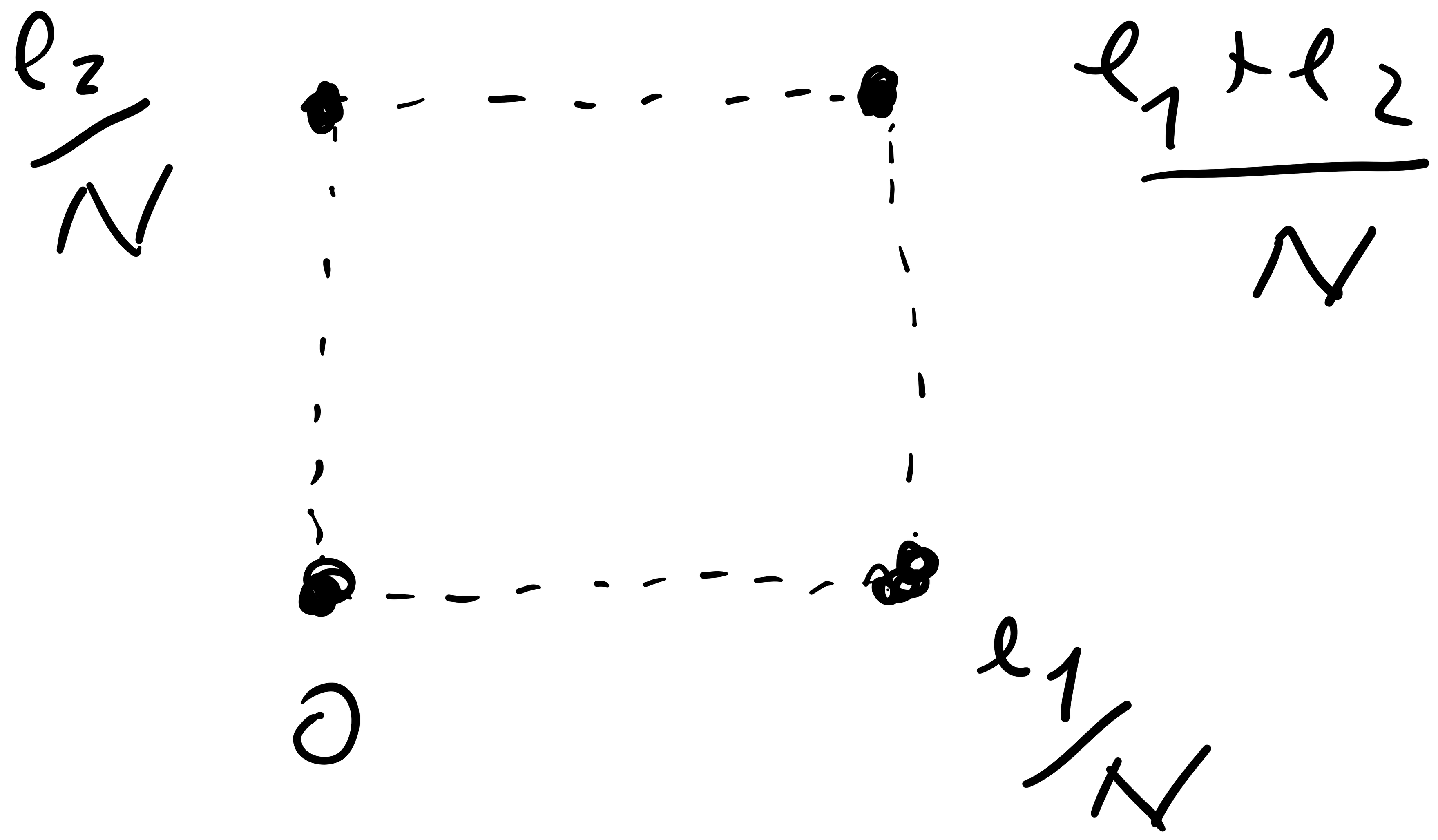
$C_i(n)$ TRANSLATIONAL INVARIANT

- OUR CLASS OF MODELS IS SUCH THAT $J^H_\eta = 0$ AN J^∇ AND J^δ ARE SUCH THAT $g(\eta)$ AND $h(\eta)$ ARE LOCAL FUNCTIONS
- $\forall \eta \qquad \nabla \cdot J^\delta_\eta(x) = 0 \qquad \forall x \in \Lambda_N$

AND THIS PART DOES NOT CONTRIBUTE TO THE EVOLUTION OF THE DENSITY

- WE HAVE $h(\eta) = -\eta(0)$ AS IN THE SEP

- $g(\eta)$ IS A LOCAL FUNCTION THAT DEPENDS JUST ON THE OCCUPATIONS OF THE LATTICE SITES

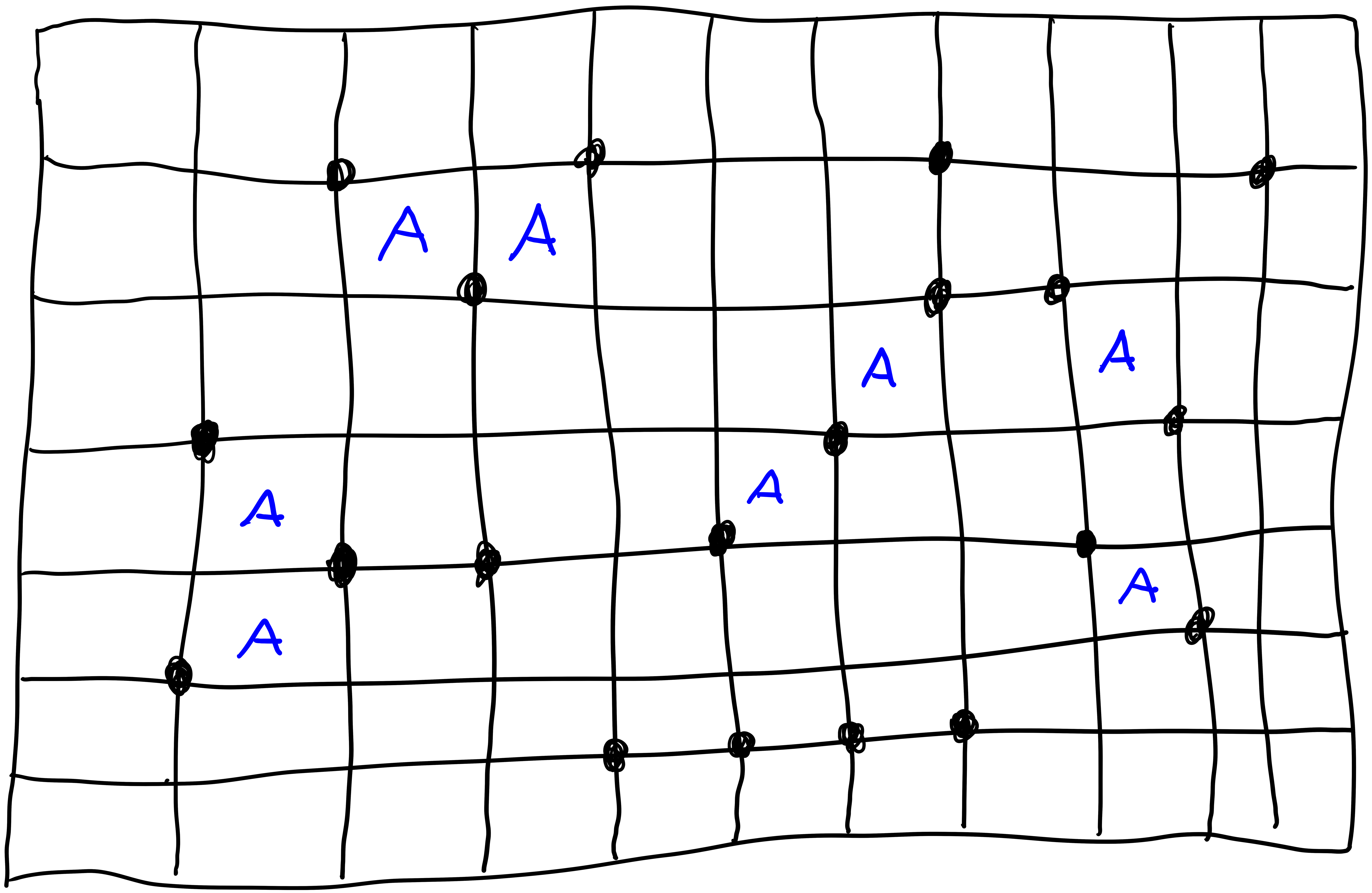


- IF $\eta = \begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}$ OR $\begin{pmatrix} \circ & \bullet \\ \bullet & \circ \end{pmatrix}$

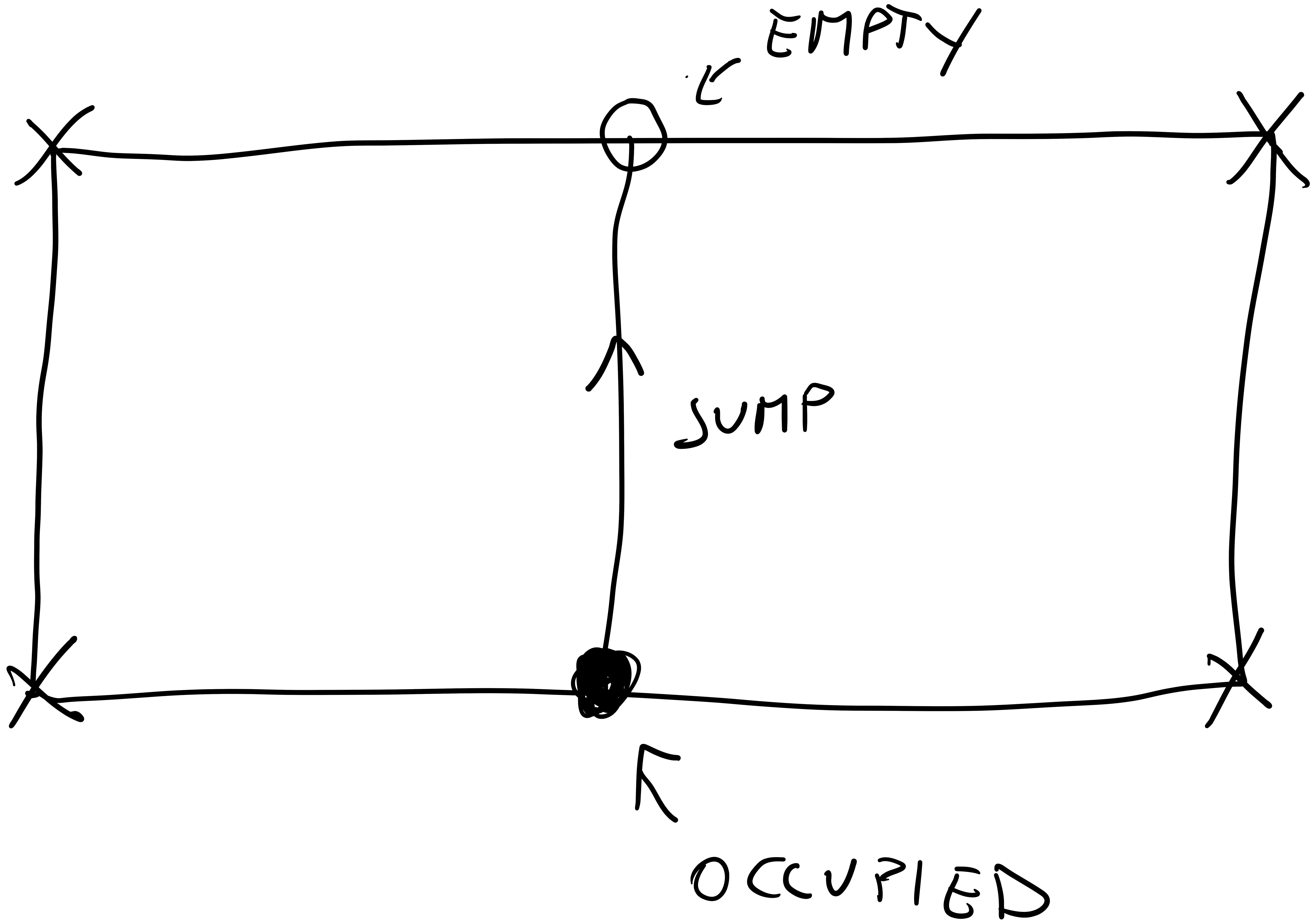
THEN $g(\eta) = \alpha$ $\quad |\alpha| < 1$ $\quad$ OTHERWISE

$$g(\eta) = 0$$

- FACES CONTAINING EXACTLY 2 PARTICLES ON OPPOSITE VERTICES ARE CALLED ACTIVATED

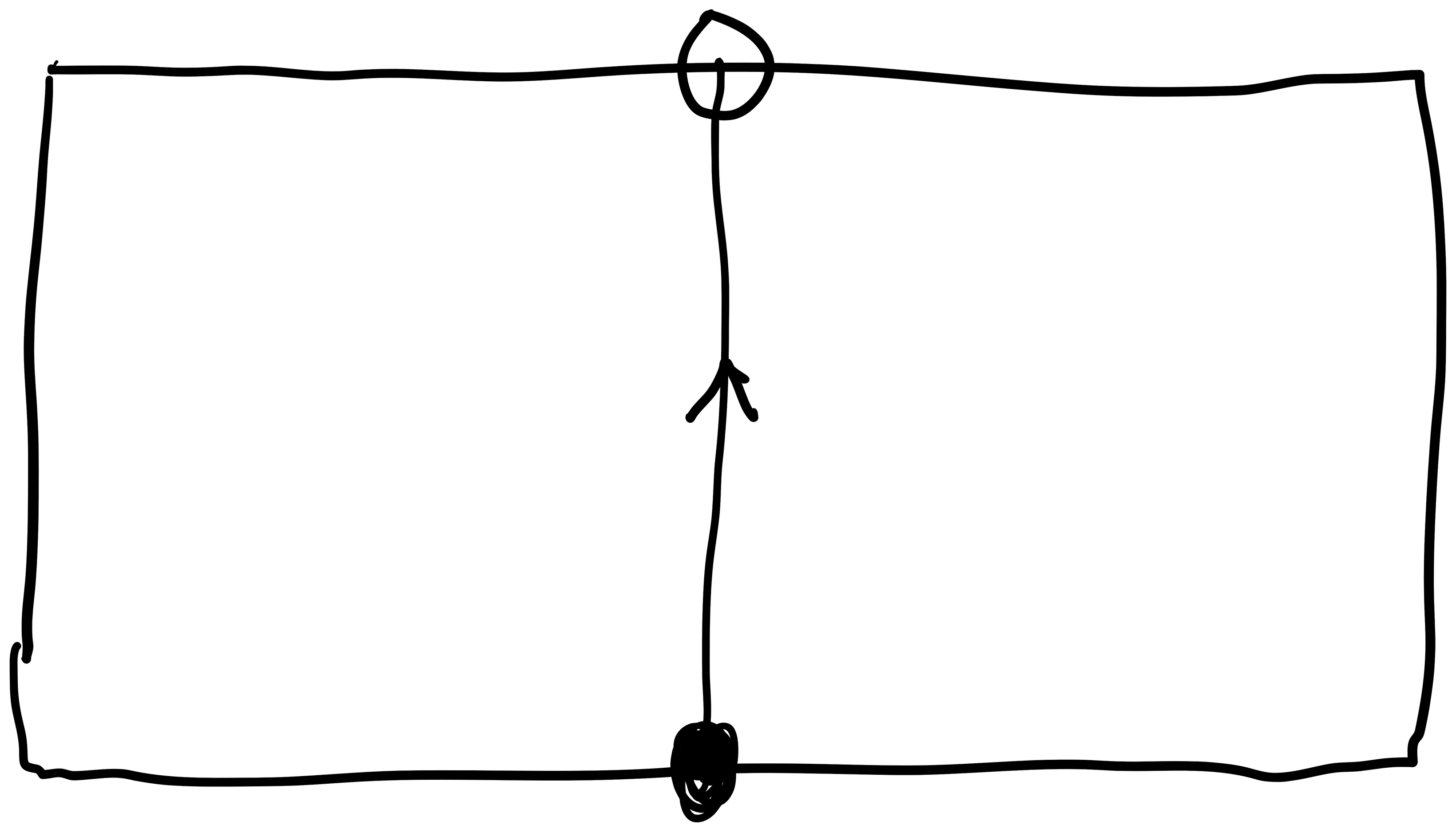


- THE RATE OF JUMP OF A PARTICLE DEPENDS ON THE OCCUPATION NUMBERS OF THE NEIGHBORING FACES



- THE JUMP RATE OF THE PARTICLE ● TO THE EMPTY SITE O DEPENDS ON THE STATES $X = \begin{cases} \bullet \\ 0 \end{cases}$

- THE VALUES OF THE RATES ARE THE FOLLOWINGS



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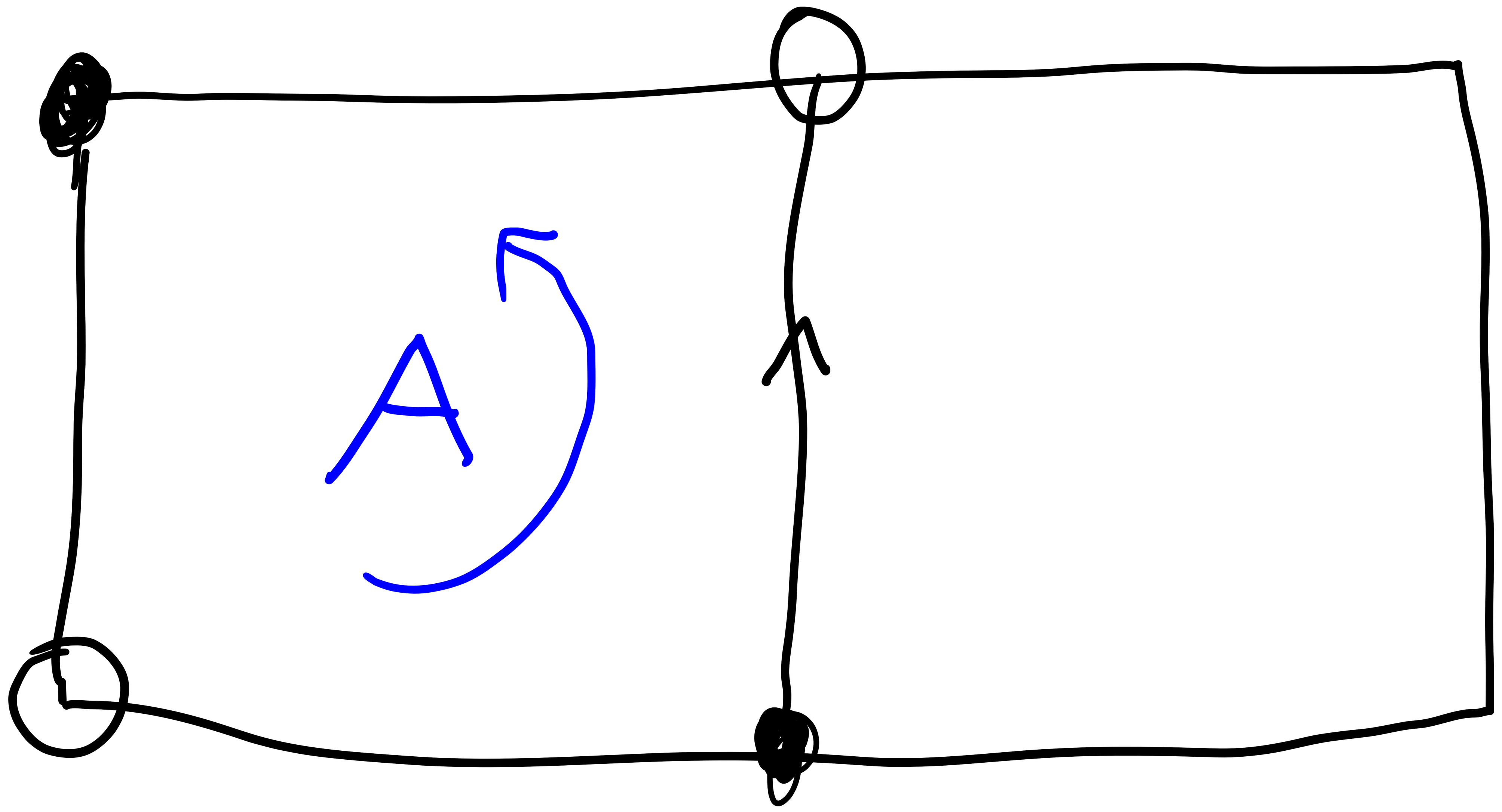
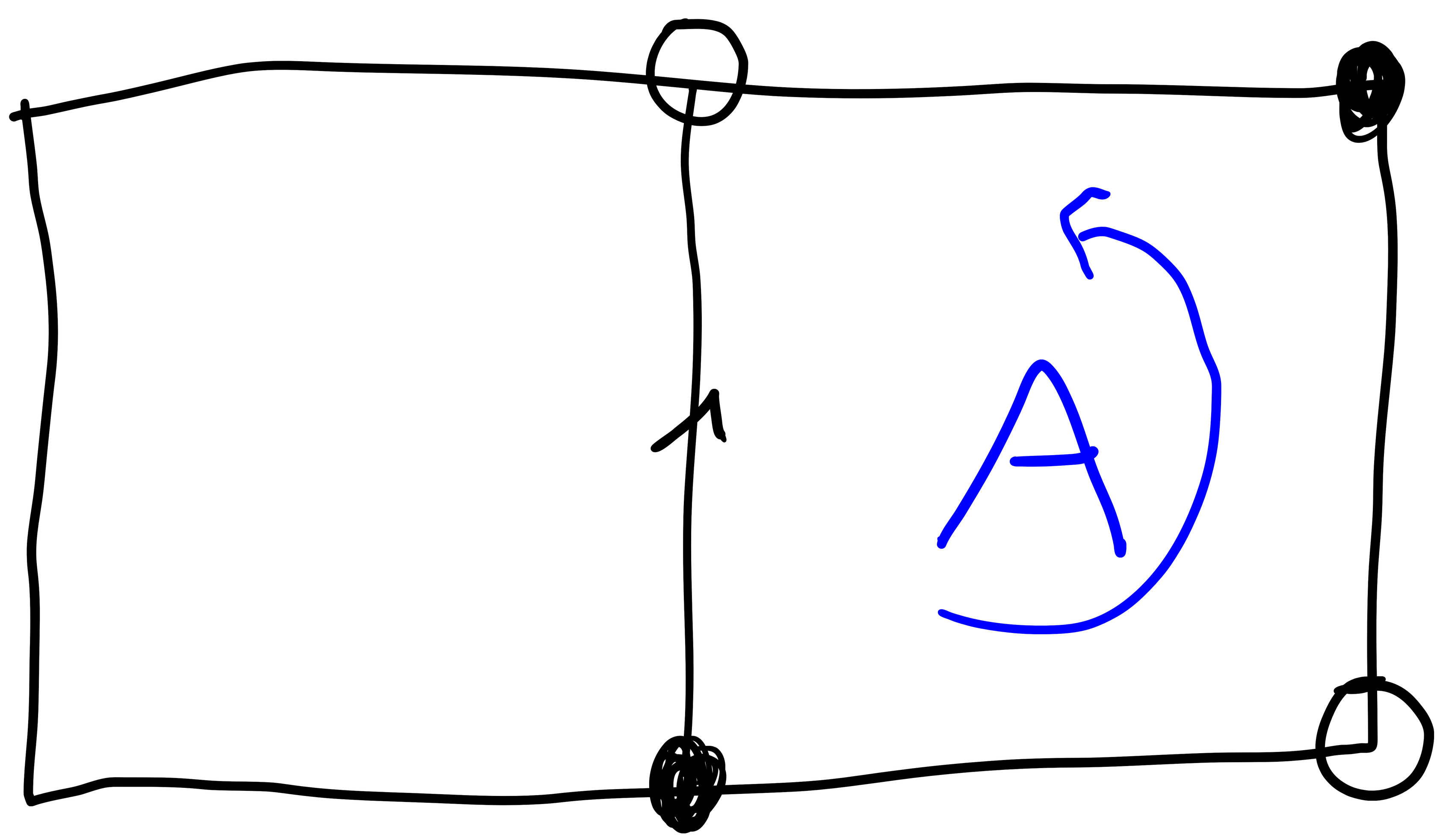
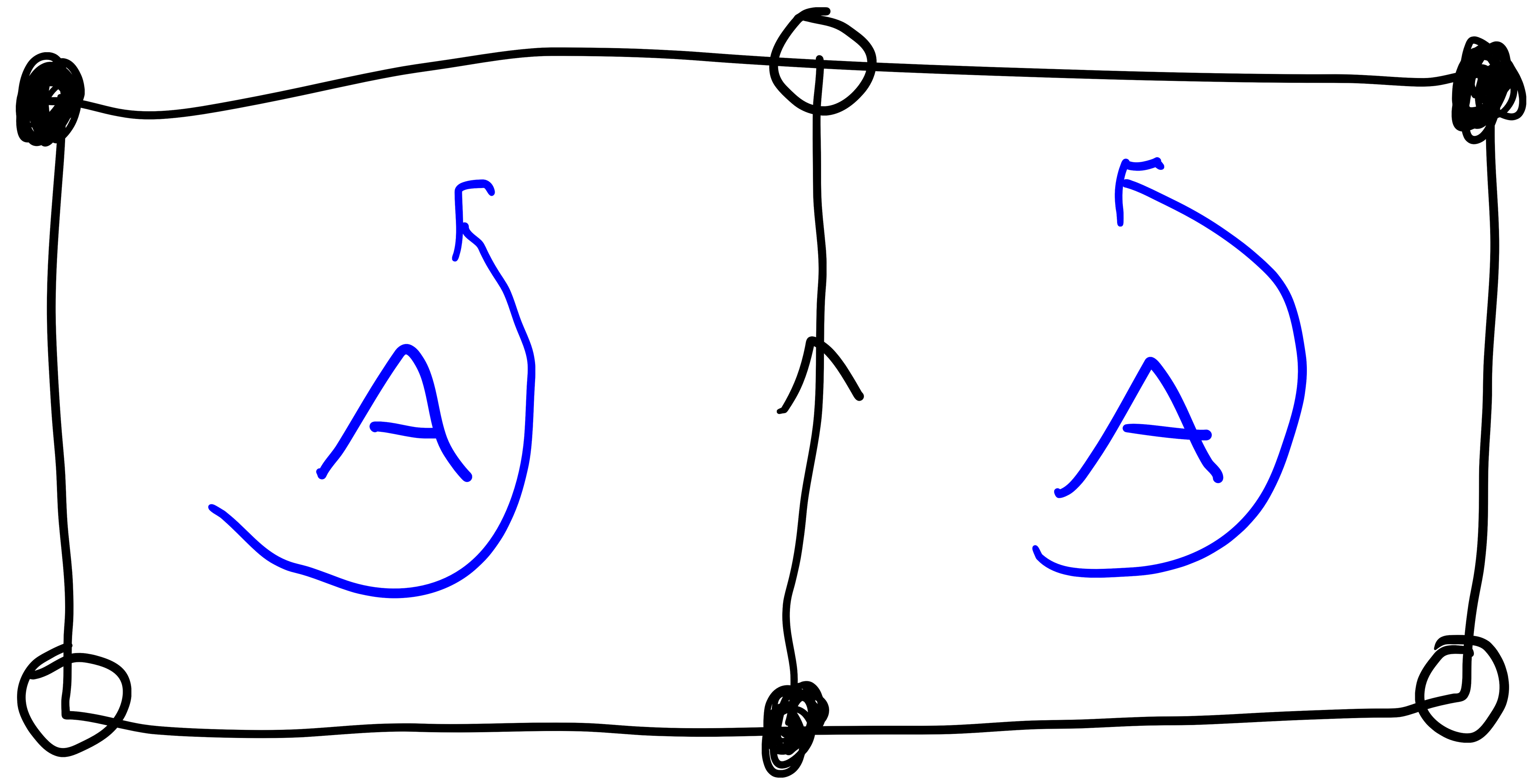


$1+\alpha$



$1-\alpha$



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- THE MODEL IS NOT REVERSIBLE, YET HAS A DIFFUSIVE BEHAVIOR

- ν_ρ BERNOULLI PRODUCT MEASURES OF ANY PARAMETER $0 \le \rho \le 1$ ARE INVARIANT.

- NOT TRIVIAL FACT TO VERIFY

$$\sum_{(x,y) \in E_N} c_{xy}(\eta) = \sum_{(x,y) \in E_N} c_{yx}(\eta^{x,y})$$

- REORGANIZE ALL TERMS CONTAINING τ_f FOR ANY GIVEN FACE f AND THEN CHECK BY DIRECT INSPECTION.

MUITO

OBRIGADO!