Approximations of the covariance operators of solutions of fractional elliptic SPDEs driven by Gaussian white noise

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Probability Webinar - IM-UFRJ
In modern applied statistics, Gaussian processes are ubiquitous. Gaussian processes indexed by multidimensional parameters are widely used in geostatistics. A common geostatistical model:

\[ Y_i = x(s_i) + \epsilon_i, \quad i = 1, \ldots, N, \quad \epsilon_i \sim N(0, \sigma^2), \]

where \( N \) is the number of observations and \( GP(m, c) \) stands for a Gaussian process with mean function \( m \) and covariance function \( c \).

Common objectives: estimate and understand the latent process given \( Y \):
- estimate model parameters \( \theta \), e.g., by \( \theta^* = \arg \max_\theta \pi(\theta|Y) \),
- use \( \pi(x(s)|Y, \theta^*) \) to answer the questions that are of interest.
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The Matérn covariance function

- Popular covariance function for random fields on $\mathbb{R}^d$:

$$c(s, s') = \frac{\sigma^2}{\Gamma(\nu)2^{\nu-1}}(\kappa\|s - s'\|)^\nu K_\nu(\kappa\|s - s'\|).$$

- $\Gamma(\cdot)$ is the Gamma function,
- $K_\nu(\cdot)$ is a modified Bessel function of the second kind,
- $\kappa > 0$ controls the correlation range and $\sigma^2$ is the variance,
- $\nu > 0$ determines the smoothness of the field.

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Whittle (1963): A Gaussian Matérn field $u(s)$ solves the SPDE

$$(\kappa^2 - \Delta)^\beta u = \mathcal{W} \quad \text{in } \mathcal{D},$$

(SP)

for Gaussian white noise $\mathcal{W}$ on $\mathcal{D} = \mathbb{R}^d$, and $4\beta = 2\nu + d$. 

Inspired by this relation, Lindgren el. al. (2011) constructed:

- computationally efficient GMRF approximations of $u(s)$,
- for bounded domains $\mathcal{D} \subseteq \mathbb{R}^d$ and $2\beta \in \mathbb{N}$,
- based on finite element discretizations of (SP).
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$^1$Lindgren, Rue and Lindstrom (2011). An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach (with discussion), JRSSB.
Idea of the SPDE approach

We will now provide a brief description of the finite element method they used.
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- To make the description simpler we will consider the nonfractional SPDE given by

\[(\kappa^2 - \Delta)u(s) = \mathcal{W}(s),\]

on some bounded domain \(D\) in \(\mathbb{R}^d\). The Laplacian operator is augmented with boundary conditions. Usually one considers Dirichle or Neumann.
Idea of the SPDE approach

The equation is interpreted in the following weak sense: for every function $\psi(s)$ from some suitable space of test functions, the following identity holds

$$\langle \psi, (\kappa^2 - \Delta)u \rangle_D \overset{d}{=} \langle \psi, W \rangle_D,$$

where $\overset{d}{=} \text{means equality in distribution and } \langle \cdot, \cdot \rangle_D \text{ is the standard inner product in } L_2(D), \langle f, g \rangle_D = \int_D f(s)g(s)ds.$
Idea of the SPDE approach

To do a finite element (FE) discretization, we will consider a finite dimensional space of test functions $V_n$. We will use a Galerkin method with $V_n = \text{span}\{\varphi_1, \ldots, \varphi_n\}$, where $\varphi_i(s), i = 1, \ldots, n$ are piecewise linear basis functions obtained from a triangulation of $\mathcal{D}$. 
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Then, we write approximate the solution $u$ by $u_n$, where $u_n$ is written in terms of the basis functions as

$$u_n(s) = \sum_{i=1}^{n} w_i \varphi_i(s).$$
Idea of the SPDE approach

We thus obtain the system of linear equations

$$\left\langle \varphi_j, (\kappa^2 - \Delta) \left( \sum_{i=1}^{n} w_i \varphi_i \right) \right\rangle_D \overset{d}{=} \left\langle \varphi_j, \mathcal{W} \right\rangle_D, \quad \text{for } j = 1, \ldots, n.$$
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\]

We begin by handling the right-hand side of the above expression. At first, notice that

\[
\langle \varphi_j, \mathcal{W} \rangle_{\mathcal{D}} = \int_{\mathcal{D}} \varphi_j(s) d\mathcal{W}(s) \sim N \left( 0, \int_{\mathcal{D}} \varphi_j^2(s) ds \right),
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since \( \varphi_j \) is deterministic.

Also, by using, again, the fact that \( \varphi_j \) is deterministic, we have that

\[
C \left( \int_D \varphi_i(s) d\mathcal{W}(s), \int_D \varphi_j(s) d\mathcal{W}(s) \right) = \int_D \varphi_i(s) \varphi_j(s) ds.
\]
Idea of the SPDE approach

This shows that

\[(\langle \varphi_1, W \rangle_D, \ldots, \langle \varphi_n, W \rangle_D) \sim N(0, C),\]

where \(C\) is an \(n \times n\) matrix with \((i,j)\)th entry given by

\[C_{i,j} = \int_D \varphi_i(s) \varphi_j(s) ds.\]

The matrix \(C\) is known as the *mass matrix* in FE theory.
Idea of the SPDE approach

Now let us handle the left hand side of the weak formulation of the SPDE. By using integration by parts we obtain for $j = 1, \ldots, n$,

$$\langle \varphi_j, (\kappa^2 - \Delta) (\sum_{i=1}^n w_i \varphi_i) \rangle_D = \sum_{i=1}^n \langle \varphi_j, (\kappa^2 - \Delta) w_i \varphi_i \rangle_D$$

$$= \sum_{i=1}^n (\kappa^2 \langle \varphi_j, \varphi_i \rangle_D + \langle \nabla \varphi_j, \nabla \varphi_i \rangle_D) w_i,$$

where the boundary terms vanish due to boundary conditions (for both Dirichlet and Neumann).
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where the boundary terms vanish due to boundary conditions (for both Dirichlet and Neumann).

We can then rewrite the last term in matrix form as

$$(\kappa^2 C + G)w,$$

where $w = (w_1, \ldots, w_n)$ and $G$ is an $n \times n$ matrix with $(i, j)$th entry given by

$$G_{i,j} = \int_D \nabla \varphi_i(s) \nabla \varphi_j(s) ds.$$
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G_{i,j} = \int_D \nabla \varphi_i(s) \nabla \varphi_j(s) ds.
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The matrix \( \mathbf{G} \) is known in FEM theory as *stiffness matrix*. 
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Putting everything together, we have that

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Therefore, \(w\) is a centered Gaussian variable with precision matrix given by
\[Q = (\kappa^2 C + G)^\top C^{-1}(\kappa^2 C + G).\]
Computational advantages of the SPDE approach

- For spatial problems, the computational cost usually scales as $O(n^{3/2})$, where $n$ is the number of basis functions. This should be compared to the $O(N^3)$ of the Gaussian random field approach.

- This implies in accurate approximations which drastically reduces the computational cost for sampling and inference.
Limitations of the SPDE approach

- $\beta$ controls the smoothness of $u(s)$, which is important for spatial prediction (Stein, 1999). Stein recommended “use the Matérn covariance” since it allows for estimating the smoothness from data.
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- The SPDE approach has the restriction that $2\beta \in \mathbb{N}$.
- Therefore, $\beta$ is typically kept fixed when the SPDE approach is used.
- $2\beta = 1.5 \notin \mathbb{N}$ corresponds to exponential covariance on $\mathbb{R}^2$. 
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In particular, by using their approach it is possible to estimate the smoothness from the data.

They also considered more general elliptic operators, thus allowing one to use the rational SPDE approach on several extensions of the SPDE approach such as

- The non-stationary Matérn models by Lindgren et al. (2011),
- The models with locally varying anisotropy by Fuglstad et al. (2015),
- The barrier models by Bakka et al. (2019).

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Assumptions of the model

Let $u$ be a Gaussian random field defined through

$$L^\beta u = \mathcal{W} \quad \text{in } \mathcal{D}, \quad \mathbb{P}\text{-a.s.},$$

(MP)

where $\beta > 0$ and

- $\mathcal{D} \subset \mathbb{R}^d$ is a bounded and convex polytope,
- $\mathcal{W}$ is Gaussian white noise on $L_2(\mathcal{D})$,
- $L$ is a linear second-order differential operator in divergence form:

$$Lu = -\nabla \cdot (H \nabla u) + \kappa^2 u,$$

with Neumann (or Dirichlet) boundary conditions.
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Assumptions on the functions \( H \) and \( \kappa \)

(I) \( H : \mathcal{D} \rightarrow \mathbb{R}^{d \times d} \) is symmetric, Lipschitz continuous and uniformly positive definite.

(II) \( \kappa \in L_\infty(\mathcal{D}) \) (with \( \text{ess inf}_{s \in \mathcal{D}} \kappa(s) \geq \kappa_0 > 0 \) in the case of Neumann boundary condition).
Spectral properties of $L$

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- Hence, there exists an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ formed by eigenvectors of $L$ whose eigenvalues are nonnegative and can be arranged in a non-decreasing order.
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- Under these hypothesis, the operator $L$ has a compact resolvent.
- Hence, there exists an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ formed by eigenvectors of $L$ whose eigenvalues are nonnegative and can be arranged in a non-decreasing order.
- By Weyl’s law, we also have that

$$\lambda_j \sim \kappa, H, \mathcal{D} j^{2/d} \quad \text{as } n \to \infty.$$
Existence and uniqueness

Proposition

Under the above assumptions and if $\beta > d/4$, (MP) has a solution which is unique $\mathbb{P}$-a.s. in $L_2(\mathcal{D})$.

Note: $\beta > d/4 \iff \nu > 0$ in the Matérn case.
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Idea: If $\mathcal{W}$ is a Gaussian white noise in $L_2(D)$ and $T : L_2(D) \to L_2(D)$ is Hilbert-Schmidt, then $T\mathcal{W}$ is a well-defined element in $L_2(\mathbb{P})$. 
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Since $L$ induces a continuous and coercive bilinear form, it is invertible.
Existence and uniqueness

**Proposition**

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Since $L$ induces a continuous and coercive bilinear form, it is invertible. Now, from Weyl’s law we have that $L^{-\beta}$ is Hilbert-Schmidt if, and only if, $\beta > d/4$. 
Regularity of trajectories

Proposition

Under the above assumptions, also if $d \in \{1, 2, 3\}$ and $2\beta \geq \gamma + d/2$, where $\gamma \in (0, 1)$, then for every $\theta \in (0, \gamma)$ the trajectories of the solution of (MP) are Hölder continuous with exponent $\theta$. 

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**Idea:** Sobolev embedding $+$ Kolmogorov-Centsov theorem.
Finite element approximation

We will consider the following assumptions related to the finite element method:
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- $V_h \subset V$ is a finite element space with continuous piecewise linear basis functions $\{\varphi_j\}_{j=1}^{n_h}$, with $n_h \in \mathbb{N}$, defined with respect to a triangulation $\mathcal{T}_h$ of the closure of the domain $\overline{D}$ indexed by the mesh width $h := \max_{T \in \mathcal{T}_h} h_T$, where $h_T := \text{diam}(T)$ is the diameter of the element $T \in \mathcal{T}_h$. 
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- The family $(\mathcal{T}_h)_{h \in (0,1)}$ of triangulations inducing the finite-dimensional subspaces $(V_h)_{h \in (0,1)}$ of $V$ is supposed to be quasi-uniform, that is, there exist constants $K_1, K_2 > 0$ such that $\rho_T \geq K_1 h_T$ and $h_T \geq K_2 h$ for all $T \in \mathcal{T}_h$ and $h \in (0,1)$. Here $\rho_T > 0$ is the radius of the largest ball inscribed in $T \in \mathcal{T}_h$. 
The discrete fractional problem

Consider for $\beta > d/4$ the discretized SPDE

$$L_h^\beta u_h = \mathcal{W}_h \quad \text{in } D \subset \mathbb{R}^d, \quad \text{(DP)}$$

where $\mathcal{W}_h$ is Gaussian white noise on $V_h$ and $L_h : V_h \to V_h$ satisfies

$$\langle L_h \psi, \phi \rangle_D = \langle H \nabla \psi, \nabla \phi \rangle_D + \langle \kappa^2 \psi, \phi \rangle_D, \quad \forall \psi, \phi \in V_h.$$
Idea of Bolin and Kirchner’s rational SPDE approach

- Construct an approximation \( u_{h,m}^R \) of the nested SPDE form

\[
P_{\ell,h} u_{h,m}^R = P_{r,h} W_h \quad \text{in } D, 
\]

with \( P_{j,h} := p_j(L_h) \) defined in terms of a polynomial \( p_j, j \in \{\ell, r\} \).

- \( P_{\ell,h} \) and \( P_{r,h} \) are commutative by construction.

⇒ Equation (RP) can be rewritten as

\[
\begin{align*}
  u_{h,m}^R &= P_{r,h} x \quad \text{in } D, \\
P_{\ell,h} x &= W_h \quad \text{in } D,
\end{align*}
\]

- Here \( x \) is a GMRF, so the model is expressed as a latent GMRF.

⇒ we can use all computational methods for GMRFs in statistics!
Construction of the polynomials

- \( p_\ell \) and \( p_r \) can be obtained from a rational approximation of \( f(x) = x^\beta \) on an interval \( J_h \) that covers the spectrum of \( L_h^{-1} \).

- To get smoothness \( m_\beta = \max\{1, \lfloor \beta \rfloor \} \), let \( \hat{f}(x) = x^{m_\beta} \hat{\hat{f}}(x) \) and compute the rational approximation \( \hat{r} \) of \( \hat{\hat{f}}(x) = x^{\beta - m_\beta} \).

- Compute \( \hat{r} \) as the \( L_\infty \)-best rational approximation of \( \hat{\hat{f}}(x) \) on \( J_h \), with polynomial orders \( m \) & \( m + 1 \) for numerator & denominator.
Convergence rates of the rational approximation

Theorem (Strong $L_2$-$L_2$ and weak convergence)

Choose $m \in \mathbb{N}$ such that $|\beta - m\beta|m \propto (\max\{\beta, 1\} \log(h))^2$. Under the above assumptions, there exists constants $C_s, C_w > 0$, independent of $h$ and $m$, such that for sufficiently small $h$,

$$\left( E[\|u - u_{h,m}^R\|_{L_2(D)}^2] \right)^{1/2} \leq C_s h^{\min\{2\beta-d/2, 2\}}$$  \hspace{1cm} (strong error),

$$|E[\varphi(u)] - E[\varphi(u_{h,m}^R)]| \leq C_w h^{\min\{4\beta-d, 2\}}$$  \hspace{1cm} (weak error),

for every $\varphi : L_2(D) \to \mathbb{R}$ sufficiently smooth (in Fréchet sense).
A covariance-based rational approximation

- For statistical applications, we are not interested in pathwise approximations, we only need an approximation of the distribution.
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- For statistical applications, we are not interested in pathwise approximations, we only need an approximation of the distribution.
- The Generalized Whittle–Matérn field is a centered Gaussian field with covariance operator $L^{-2\beta}$.

\[
L^{-2\beta}h = (L^{-1}h)^{2\beta} \approx L^{-\tilde{m}\beta}h p_{\ell}(Lh)p_{r}(Lh) - 1
\]

where the polynomials $p_{\ell}$ and $p_{r}$ are obtained from a rational approximation of $f(x) = x^{2\beta} - \tilde{m}\beta$ and $\tilde{m}\beta = \lfloor \beta \rfloor$. 
For statistical applications, we are not interested in pathwise approximations, we only need an approximation of the distribution.

The Generalized Whittle–Matérn field is a centered Gaussian field with covariance operator $L^{-2\beta}$.

We can perform the rational approximation of the covariance operator:

$$L_h^{-2\beta} = (L_h^{-1})^{2\beta} \approx L_h^{-\tilde{m}_\beta} p_\ell(L_h) p_r(L_h)^{-1}$$

where the polynomials $p_\ell$ and $p_r$ are obtained from a rational approximation of $f(x) = x^{2\beta - \tilde{m}_\beta}$ and $\tilde{m}_\beta = \lfloor \beta \rfloor$. 
Theorem
Let \( r_{\beta,m}(L_h) = L_h^{-\tilde{m}_\beta} p_\ell(L_h)p_r(L_h)^{-1} \), where \( m \) indicates the degree of the polynomial in the numerator of the rational approximation. Under all the previous assumptions, we have that for every \( \beta > d/4 \) and every \( \varepsilon > 0 \) there exists \( m \in \mathbb{N} \) such that

\[
\| L^{-2\beta} - r_{\beta,m}(L_h) \|_{L^2(L_2(D))} \leq h^{\min\{4\beta-d/2-\varepsilon,2\}}.
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\]

Idea: The idea is based on Cox and Kirchner’s (2020) paper. There they obtained some bounds on operator’s norm, and we use several times the inequality

\[
\|AB\|_{HS} \leq \|A\| \|B\|_{HS}
\]

and

\[
\|AB\|_{HS} \leq \|A\|_{HS} \|B\|.
\]

Where we show that the Hilbert-Schmidt norm appearing are bounded and the rate comes from the bounds in the operator’s norm.
Consequences

For an $m$-order approximation for the operator-based method, we can here choose the order as $2(m + m_\beta) - \tilde{m}_\beta$ to have the same computational cost.

This also enables us to represent the rational approximation $u_{h,m}^C \sim N(0, L_h^{-\tilde{m}_\beta} p_\ell(L_h)p_r(L_h)^{-1})$ as

$$u_{h,m}^C \overset{d}{=} \sum_{k=1}^m u_{r,k}, \quad u_{r,k} \sim N(0, a_k L_h^{-\tilde{m}_\beta} (L_h + b_k I)^{-1})$$

where the $u_{r,k}$ are independent and $a_k, b_k > 0$. 
In terms of the finite element representation, the covariance matrix is given by

\[ \Sigma_{\tilde{u}} = (L^{-1}C)^{\tilde{m}_\beta} \sum_{k=1}^{m} a_k (L + b_k C)^{-1} + K \]

where:

\[ K = \begin{cases} 
  k C & \tilde{m}_\beta = 0 \\
  k L^{-1}(C L^{-1})^{\tilde{m}_\beta-1} & \tilde{m}_\beta \geq 1 
\end{cases} \]

and \( L \) is the matrix of the operator \( L_h \) in terms of the basis functions of \( V_h \).
In order to perform statistical inference, one needs to invert the mass matrix $C$, which is not diagonal.
Improving computational performance

- In order to perform statistical inference, one needs to invert the mass matrix $C$, which is not diagonal.
- To circumvent this problem one replaces $C$ by a diagonal matrix $\tilde{C}$ with $i$th diagonal entry given by the sum of the elements in the $i$th row.

This was called the "Markov approximation" in the paper by Lindgren et al. (2011). In the numerical analysis literature this is known as the lumped mass method.
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- It is known in numerical analysis that in our finite element setup the replacement of the mass matrix $C$ by the lumped mass matrix $\tilde{C}$ is equivalent to replace the $L_2$ inner product by a quadrature approximation.
- We will then use some tools from numerical analysis to obtain our rate of convergence.
The lumped mass method

Fix some $h \in (0, 1)$ and an element $\tau \in \mathcal{T}_h$. Let $V_{1,\tau}, \ldots, V_{3,\tau}$ be the vertices of the triangle $\tau$ and consider the quadrature formula

$$Q_{\tau,h}(f) = \frac{\text{area}(\tau)}{3} \sum_{j=1}^{3} f(V_{j,\tau}).$$

Obs: The quadrature scheme given above is exact for polynomials of degree less or equal to 1.

Consider the following quadrature scheme to approximate the $L^2$-inner-product:

$$\langle f, g \rangle_h = \sum_{\tau \in \mathcal{T}_h} Q_{\tau,h}(fg).$$
The lumped mass method

Take \( \{ \varphi_i \}_{i=1}^{n_h} \) to be the set of standard basis of \( V_h \) consisting of the “hat” basis functions, which are continuous and piecewise linear, defined with respect to the triangulation \( T_h \) in such a way that if \( \{ V_i \}_{i=1}^{n_h} \) are the vertices of the triangulation \( T_h \), then \( \varphi_i(V_j) = \delta_{ij} \). Then, we have that for every \( j = 1, \ldots, n_h \)

\[
\| \varphi_j \|_h^2 = \langle \varphi_j, \varphi_j \rangle_h = \sum_{k=1}^{n_h} \langle \varphi_j, \varphi_k \rangle_{L^2(D)}
\]

and that

\[
\langle \varphi_i, \varphi_j \rangle_h = 0
\]

if \( i \neq j \) as \( \varphi_i(x) \varphi_j(x) \) vanishes at all vertices of \( T_h \) (see Thomee, chapter 15 and also to Jin, Lazarov and Zhou (2013)) for further details.
Define the operator $\tilde{L}_h : \tilde{V}_h \to \tilde{V}_h$ as

$$\langle \tilde{L}_h \phi_h, \psi_h \rangle_h = a_h(\phi_h, \psi_h) = \langle H \nabla \phi_h, \nabla \psi_h \rangle_h + \langle \kappa^2 \phi_h, \psi_h \rangle_h.$$

By defining the operator in this way, for constant $\kappa$ one will replace every mass matrix $C$ by the lumped mass matrix $\tilde{C}$, even the mass matrices inside the discretization of the operator.
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By defining the operator in this way, for constant \( \kappa \) one will replace every mass matrix \( C \) by the lumped mass matrix \( \tilde{C} \), even the mass matrices inside the discretization of the operator.

One can also define the operator \( \tilde{L}_h : \tilde{V}_h \to \tilde{V}_h \) as
\[
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In this case we are only replacing the mass matrices "outside" of the operator.
We have the following theorem:

**Theorem**

*Under all the previous assumptions and additionally, if* $\beta > 1/2$ *and* $H$, $\kappa$ *belong to* $W^{2,\infty}(D)$, *then,*

$$\| (L_h^{-2\beta} - \tilde{L}_h^{-2\beta}) \Pi_h \|_{L(H^1(D))} \leq C h^2.$$  

**Obs:** If we consider the second form of discretization, the above rate hold under the usual assumptions on $H$ and $\kappa$. 


Rate of convergence of the lumped mass operator in the Hilbert-Schmidt norm

Theorem

Under all the previous assumptions and additionally, if $\beta > 1$ and $H, \kappa$ belong to $W^{2,\infty}(\mathcal{D})$, then,

$$
\|(L_h^{-2\beta} - \tilde{L}_h^{-2\beta})\Pi_h\|_{L_2(H^1(\mathcal{D}))} \leq C h^2.
$$

The idea of the proof is to use sharp rates of the quadrature approximation of the $L_2$-inner product with the strategy used for the covariance-based rational approximation.
We now move our attention towards maximum likelihood estimation of the parameters of the model. To this end, consider the SPDE:

$$\tau (\kappa^2 I - \Delta)^\beta u = \mathcal{W},$$

where $\tau, \kappa$ and $\beta$ are assumed to be positive constants.
Identifiability

We have the following theorem by Bolin and Kirchner (2020):

**Theorem**

Let $\mathcal{D} \subset \mathbb{R}^d$ be bounded, open and connected. For $i \in \{1, 2\}$, let $\beta_i > d/4$, $\kappa_i, \tau_i > 0$, and consider the Gaussian measure

$$\mu_i := N(m_i, Q_i^{-1})$$

on $L_2(\mathcal{D})$ with mean $m_i = 0$ and precision operator

$$Q_i = \tau_i^2 L_i^{2\beta_i}$$

where for $i \in \{1, 2\}$, the operators $L_i = \kappa_i^2 I - \Delta$ are augmented with the same homogeneous Neumann or Dirichlet boundary conditions. Then, $\mu_1$ and $\mu_2$ are equivalent if, and only if, $\beta_1 = \beta_2$ and $\tau_1 = \tau_2$. 
Consistency of the MLE of $\tau$

In the first scenario, assume that we have a sequence of finite sample points $\mathcal{D}_n$ whose union is dense in $\mathcal{D}$. Then,

**Proposition**

Assume $\beta > d/4$ is known, and let $\hat{\tau}_n^2$ be the MLE of $\tau$ based on the sample $u(s_1), \ldots, u(s_{k_n})$, where $s_i \in \mathcal{D}_n$ and $u$ is the solution of

$$\tau(\kappa^2 I - \Delta)^\beta u = \mathcal{W}.$$ 

Then, $\hat{\tau}_n^2$ is weakly consistent.
Consistency of the MLE of $\tau$

In the second scenario, we also assume we have a sequence of finite sample points $\mathcal{D}_n$ whose union is dense in $\mathcal{D}$. We consider the same equation, we also consider a finite element approximation of $u$, given by

$$u_{n_h}(s) = \sum_{i=1}^{n_h} w_i \varphi_i(s).$$

Let $A$ be an $n \times n_h$ matrix with $i,j$th entry given by $\varphi_j(s_i)$. Then, we assume we observe $y = (y_1, \ldots, y_n)$, where

$$y = Aw.$$

**Proposition**

Assume $\beta > d/4$ is known, and let $h := h(n)$ be any sequence such that $n \leq n_h$ for every $n$. Let $\hat{\tau}_n^2$ be the MLE of $\tau$ based on the sample $y_1, \ldots, y_n$. Then, $\hat{\tau}_n^2$ is weakly consistent.
Consistency of the MLE of $\tau$

Idea: Combine the identifiability with an explicit expression of the maximum likelihood estimator and the law of large numbers.
Finally, in the third scenario we replace, in the likelihood equation, the mass matrix by the lumped mass matrix. Then, we have the following Proposition:

**Proposition**

Consider the same assumptions of the second scenario. Additionally, assume \( d = 2 \) and \( \beta > 1/2 \) is known, and let \( h = h(n) \) be chosen such that \( nh^2 = o(1) \) and \( n \leq n_h \) for every \( n \). Let \( \hat{\tau}^2_{n,lump} \) be the MLE of \( \tau \) based on the sample \( y_1, \ldots, y_n \) with the mass matrix replaced by the lumped mass matrix. Then, \( \hat{\tau}^2_{n,lump} \) is weakly consistent.

**Idea of the proof:** Use the bound in operator’s norm for the lumped mass covariance operator to bound the difference \( |\hat{\tau}^2_{n,lump} - \hat{\tau}^2_n| \).
Thank you!