Approximations of the covariance operators of solutions of fractional elliptic SPDEs driven by Gaussian white noise

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Joint work with David Bolin
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Probability Webinar - IM-UFRJ

Lantent Gaussian models and random fields

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- Gaussian processes indexed by multidimensional parameters are widely used in geostatistics.
- A common geostatistical model:

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\begin{aligned}
Y_i = x(\mathbf{s}_i) + \varepsilon_i, \quad i = 1, \ldots, N, \quad \varepsilon_i \sim \mathsf{N}(0, \sigma^2), \\
x(\mathbf{s}) \sim \mathsf{GP}\left(m(\mathbf{s}),\, c(\mathbf{s}, \mathbf{s}')\right),
\end{aligned}
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  where N is the number of observations and $GP(m, c)$ stands for a Gaussian process with mean function m and covariance function c.

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  where N is the number of observations and $GP(m, c)$ stands for a Gaussian process with mean function m and covariance function c.
- Common objectives: estimate and understand the latent process given $\mathbf{Y}$:
  - estimate model parameters $\boldsymbol{\theta}$, e.g., by $\boldsymbol{\theta}^* = \arg\max_{\boldsymbol{\theta}} \pi(\boldsymbol{\theta}|\mathbf{Y})$,
  - use $\pi(x(s)|\mathbf{Y}, \boldsymbol{\theta}^*)$ to answer the questions that are of interest.

The Matérn covariance function

- Popular covariance function for random fields on $\mathbb{R}^d$:

$$c(\mathbf{s}, \mathbf{s}') = \frac{\sigma^2}{\Gamma(\nu)2^{\nu-1}}(\kappa\|\mathbf{s} - \mathbf{s}'\|)^\nu K_\nu(\kappa\|\mathbf{s} - \mathbf{s}'\|).$$

  - $\Gamma(\cdot)$ is the Gamma function,
  - $K_\nu(\cdot)$ is a modified Bessel function of the second kind,
  - $\kappa > 0$ controls the correlation range and σ^2 is the variance,
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- Unlike other popular covariance functions, the Matérn class has a parameter that controls the smoothness of the process.
- **Main drawback of this approach:** The computational time needed in order to perform statistical inference usually scales as $\mathcal{O}(N^3)$.

The SPDE approach

- Whittle (1963): A Gaussian Matérn field $u(\mathrm{s})$ solves the SPDE

$$(\kappa^2 - \Delta)^{\beta} u = \mathcal{W} \quad \text{in } \mathcal{D}, \tag{SP}$$

for Gaussian white noise $\mathcal{W}$ on $\mathcal{D} = \mathbb{R}^d$, and $4\beta = 2\nu + d$.

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  for Gaussian white noise $\mathcal{W}$ on $\mathcal{D} = \mathbb{R}^d$, and $4\beta = 2\nu + d$.
- Inspired by this relation, Lindgren el. al. (2011)[1] constructed:
  - computationally efficient GMRF approximations of $u(\mathbf{s})$,
  - for bounded domains $\mathcal{D} \subsetneq \mathbb{R}^d$ and $2\beta \in \mathbb{N}$,

  based on finite element discretizations of (SP).

[1] Lindgren, Rue and Lindstrom (2011). An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach (with discussion), JRSSB.

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- To make the description simpler we will consider the nonfractional SPDE given by
$$(\kappa^2 - \Delta)u(\mathbf{s}) = \mathcal{W}(\mathbf{s}),$$
on some bounded domain $\mathcal{D}$ in $\mathbb{R}^d$. The Laplacian operator is augmented with boundary conditions. Usually one considers Dirichle or Neumann.

Idea of the SPDE approach

The equation is interpreted in the following weak sense:
for every function $\psi(\mathbf{s})$ from some suitable space of test functions, the following identity holds

$$\langle \psi, (\kappa^2 - \Delta)u \rangle_{\mathcal{D}} \stackrel{d}{=} \langle \psi, \mathcal{W} \rangle_{\mathcal{D}},$$

where $\stackrel{d}{=}$ means equality in distribution and $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ is the standard inner product in $L_2(\mathcal{D})$, $\langle f, g \rangle_{\mathcal{D}} = \int_{\mathcal{D}} f(\mathbf{s}) g(\mathbf{s}) d\mathbf{s}$.

Idea of the SPDE approach

To do a finite element (FE) discretization, we will consider a finite dimensional space of test functions V_n. We will use a Galerkin method with $V_n = \text{span}\{\varphi_1, \ldots, \varphi_n\}$, where $\varphi_i(\mathbf{s}), i = 1, \ldots, n$ are piecewise linear basis functions obtained from a triangulation of $\mathcal{D}$.

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Then, we write approximate the solution u by u_n, where u_n is written in terms of the basis functions as

$$u_n(\mathbf{s}) = \sum_{i=1}^{n} w_i \varphi_i(\mathbf{s}).$$

Idea of the SPDE approach

We thus obtain the system of linear equations

$$\left\langle \varphi_j, (\kappa^2 - \Delta)\left(\sum_{i=1}^{n} w_i \varphi_i\right)\right\rangle_{\mathcal{D}} \stackrel{d}{=} \langle \varphi_j, \mathcal{W}\rangle_{\mathcal{D}}, \quad \text{for } j = 1, \ldots, n.$$

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We begin by handling the right-hand side of the above expression. At first, notice that

$$\langle \varphi_j, \mathcal{W} \rangle_{\mathcal{D}} = \int_{\mathcal{D}} \varphi_j(\mathbf{s}) d\mathcal{W}(\mathbf{s}) \sim N\left(0, \int_{\mathcal{D}} \varphi_j^2(\mathbf{s}) d\mathbf{s}\right),$$

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Also, by using, again, the fact that φ_j is deterministic, we have that

$$C\left(\int_{\mathcal{D}} \varphi_i(\mathbf{s}) d\mathcal{W}(\mathbf{s}), \int_{\mathcal{D}} \varphi_j(\mathbf{s}) d\mathcal{W}(\mathbf{s})\right) = \int_{\mathcal{D}} \varphi_i(\mathbf{s})\varphi_j(\mathbf{s}) d\mathbf{s}.$$

Idea of the SPDE approach

This shows that

$$(\langle \varphi_1, \mathcal{W} \rangle_{\mathcal{D}}, \ldots, \langle \varphi_n, \mathcal{W} \rangle_{\mathcal{D}}) \sim N(0, \mathbf{C}),$$

where $\mathbf{C}$ is an $n \times n$ matrix with (i, j)th entry given by

$$\mathbf{C}_{i,j} = \int_{\mathcal{D}} \varphi_i(\mathbf{s}) \varphi_j(\mathbf{s}) d\mathbf{s}.$$

The matrix $\mathbf{C}$ is known as the *mass matrix* in FE theory.

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Now let us handle the left hand side of the weak formulation of the SPDE. By using integration by parts we obtain for $j = 1, \ldots, n$,

$$\begin{aligned} \left\langle \varphi_j, (\kappa^2 - \Delta) \left(\sum_{i=1}^n w_i \varphi_i \right) \right\rangle_{\mathcal{D}} &= \sum_{i=1}^n \langle \varphi_j, (\kappa^2 - \Delta) w_i \varphi_i \rangle_{\mathcal{D}} \\ &= \sum_{i=1}^n (\kappa^2 \langle \varphi_j, \varphi_i \rangle_{\mathcal{D}} + \langle \nabla \varphi_j, \nabla \varphi_i \rangle_{\mathcal{D}}) w_i, \end{aligned}$$

where the boundary terms vanish due to boundary conditions (for both Dirichlet and Neumann).

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where the boundary terms vanish due to boundary conditions (for both Dirichlet and Neumann).

We can then rewrite the last term in matrix form as

$$(\kappa^2 \mathbf{C} + \mathbf{G}) \mathbf{w},$$

where $\mathbf{w} = (w_1, \dots, w_n)$ and $\mathbf{G}$ is an $n \times n$ matrix with (i, j)th entry given by

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The matrix $\mathbf{G}$ is known in FEM theory as *stiffness matrix*.

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Therefore, $\mathbf{w}$ is a centered Gaussian variable with precision matrix given by

$$\mathbf{Q} = (\kappa^2 \mathbf{C} + \mathbf{G})^\top \mathbf{C}^{-1} (\kappa^2 \mathbf{C} + \mathbf{G}).$$

Computational advantages of the SPDE approach

- For spatial problems, the computational cost usually scales as $\mathcal{O}(n^{3/2})$, where n is the number of basis functions. This should be compared to the $\mathcal{O}(N^3)$ of the Gaussian random field approach.
- This implies in accurate approximations which drastically reduces the computational cost for sampling and inference.

Limitations of the SPDE approach

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- The SPDE approach has the restriction that $2\beta \in \mathbb{N}$.
- Therefore, β is typically kept fixed when the SPDE approach is used.
- $2\beta = 1.5 \notin \mathbb{N}$ corresponds to exponential covariance on $\mathbb{R}^2$.

The rational SPDE approach

- Bolin and Kirchner (2020) [2] introduced the rational SPDE approach, which allows one to consider arbitrary smoothness.

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- Bolin and Kirchner (2020) [2] introduced the rational SPDE approach, which allows one to consider arbitrary smoothness.
- In particular, by using their approach it is possible to estimate the smoothness from the data.
- They also considered more general elliptic operators, thus allowing one to use the rational SPDE approach on several extensions of the SPDE approach such as
  - The non-stationary Matérn models by Lindgren et al. (2011),
  - The models with locally varying anisotropy by Fuglstad et al. (2015),
  - The barrier models by Bakka et al. (2019).

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Assumptions of the model

Let u be a Gaussian random field defined through

$$L^\beta u = \mathcal{W} \quad \text{in } \mathcal{D}, \quad \mathbb{P}\text{-a.s.,} \qquad \text{(MP)}$$

where $\beta > 0$ and

- $\mathcal{D} \subset \mathbb{R}^d$ is a bounded and convex polytope,
- $\mathcal{W}$ is Gaussian white noise on $L_2(\mathcal{D})$,
- L is a linear second-order differential operator in divergence form:

$$Lu = -\nabla \cdot (\mathbf{H}\nabla u) + \kappa^2 u,$$

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Assumptions on the functions $\mathbf{H}$ and κ

(I) $\mathbf{H} \colon \mathcal{D} \to \mathbb{R}^{d \times d}$ is symmetric, Lipschitz continuous and uniformly positive definite.

(II) $\kappa \in L_\infty(\mathcal{D})$ (with $\operatorname{ess\,inf}_{\mathbf{s}\in\mathcal{D}} \kappa(\mathbf{s}) \geq \kappa_0 > 0$ in the case of Neumann boundary condition).

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- Hence, there exists an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ formed by eigenvectors of L whose eigenvalues are nonnegative and can be arranged in a non-decreasing order.
- By Weyl's law, we also have that

$$\lambda_j \eqsim_{\kappa,\boldsymbol{H},\mathcal{D}} j^{2/d} \quad \text{as } n \to \infty.$$

Existence and uniqueness

Proposition

Under the above assumptions and if $\beta > d/4$*, (MP) has a solution which is unique* $\mathbb{P}$*-a.s. in* $L_2(\mathcal{D})$*.*

Note: $\beta > d/4 \Leftrightarrow \nu > 0$ in the Matérn case.

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Idea: If $\mathcal{W}$ is a Gaussian white noise in $L_2(\mathcal{D})$ and $T : L_2(\mathcal{D}) \to L_2(\mathcal{D})$ is Hilbert-Schmidt, then $T\mathcal{W}$ is a well-defined element in $L_2(\mathbb{P})$.

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Now, from Weyl's law we have that $L^{-\beta}$ is Hilbert-Schmidt if, and only if, $\beta > d/4$.

Regularity of trajectories

Proposition

Under the above assumptions, also if $d \in \{1, 2, 3\}$ *and* $2\beta \geq \gamma + d/2$, *where* $\gamma \in (0, 1)$, *then for every* $\theta \in (0, \gamma)$ *the trajectories of the solution of (MP) are Hölder continuous with exponent* θ.

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Idea: Sobolev embedding + Kolmogorov-Centsov theorem.

Finite element approximation

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- $V_h \subset V$ is a finite element space with continuous piecewise linear basis functions $\{\varphi_j\}_{j=1}^{n_h}$, with $n_h \in \mathbb{N}$, defined with respect to a triangulation $\mathcal{T}_h$ of the closure of the domain $\overline{\mathcal{D}}$ indexed by the mesh width $h := \max_{T \in \mathcal{T}_h} h_T$, where $h_T := diam(T)$ is the diameter of the element $T \in \mathcal{T}_h$.

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- The family $(\mathcal{T}_h)_{h\in(0,1)}$ of triangulations inducing the finite-dimensional subspaces $(V_h)_{h\in(0,1)}$ of V is supposed to be quasi-uniform, that is, there exist constants $K_1, K_2 > 0$ such that $\rho_T \geq K_1 h_T$ and $h_T \geq K_2 h$ for all $T \in \mathcal{T}_h$ and $h \in (0,1)$. Here $\rho_T > 0$ is the radius of the largest ball inscribed in $T \in \mathcal{T}_h$.

The discrete fractional problem

Consider for $\beta > d/4$ the discretized SPDE

$$L_h^{\beta} u_h = \mathcal{W}_h \quad \text{in } \mathcal{D} \subset \mathbb{R}^d, \tag{DP}$$

where $\mathcal{W}_h$ is Gaussian white noise on V_h and $L_h \colon V_h \to V_h$ satisfies

$$\langle L_h \psi, \phi \rangle_{\mathcal{D}} = \langle \mathbf{H} \nabla \psi, \nabla \phi \rangle_{\mathcal{D}} + \langle \kappa^2 \psi, \phi \rangle_{\mathcal{D}}, \quad \forall \psi, \phi \in V_h.$$

Idea of Bolin and Kirchner's rational SPDE approach

- Construct an approximation $u_{h,m}^R$ of the nested SPDE form

$$P_{\ell,h} u_{h,m}^R = P_{r,h} \mathcal{W}_h \quad \text{in } \mathcal{D}, \tag{RP}$$

  with $P_{j,h} := p_j(L_h)$ defined in terms of a polynomial p_j, $j \in \{\ell, r\}$.
- $P_{\ell,h}$ and $P_{r,h}$ are commutative by construction.

$\Rightarrow$ Equation (RP) can be rewritten as

$$\begin{aligned} u_{h,m}^R &= P_{r,h} x \quad \text{in } \mathcal{D}, \\ P_{\ell,h} x &= \mathcal{W}_h \qquad \text{in } \mathcal{D}, \end{aligned}$$

- Here x is a GMRF, so the model is expressed as a latent GMRF.

$\Rightarrow$ we can use all computational methods for GMRFs in statistics!

Construction of the polynomials

- p_ℓ and p_r can be obtained from a rational approximation of $f(x) = x^\beta$ on an interval J_h that covers the spectrum of L_h^{-1}.
- To get smoothness $m_\beta = \max\{1, \lfloor\beta\rfloor\}$, let $f(x) = x^{m_\beta}\hat{f}(x)$ and compute the rational approximation $\hat{r}$ of $\hat{f}(x) = x^{\beta - m_\beta}$.
- Compute $\hat{r}$ as the L_∞-best rational approximation of $\hat{f}(x)$ on J_h, with polynomial orders m & $m+1$ for numerator & denominator.

Convergence rates of the rational approximation

Theorem (Strong L_2-L_2 and weak convergence)

Choose $m \in \mathbb{N}$ *such that* $|\beta - m_\beta| m \propto (\max\{\beta, 1\} \log(h))^2$. *Under the above assumptions, there exists constants* $C_s, C_w > 0$, *independent of* h *and* m, *such that for sufficiently small* h,

$$\left(E[\|u - u_{h,m}^R\|^2_{L_2(\mathcal{D})}]\right)^{1/2} \leq C_s h^{\min\{2\beta - d/2,\, 2\}} \qquad \textit{(strong error)},$$

$$|E[\varphi(u)] - E[\varphi(u_{h,m}^R)]| \leq C_w h^{\min\{4\beta - d,\, 2\}} \qquad \textit{(weak error)},$$

for every $\varphi : L_2(\mathcal{D}) \to \mathbb{R}$ *sufficiently smooth (in Fréchet sense).*

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- For statistical applications, we are not interested in pathwise approximations, we only need an approximation of the distribution.
- The Generalized Whittle–Matérn field is a centered Gaussian field with covariance operator $L^{-2\beta}$.
- We can perform the rational approximation of the covariance operator:

$$L_h^{-2\beta} = (L_h^{-1})^{2\beta} \approx L_h^{-\widetilde{m}_\beta} p_\ell(L_h) p_r(L_h)^{-1}$$

where the polynomials p_ℓ and p_r are obtained from a rational approximation of $f(x) = x^{2\beta - \widetilde{m}_\beta}$ and $\widetilde{m}_\beta = \lfloor \beta \rfloor$.

Convergence rates for the covariance-based approximation

Theorem

Let $r_{\beta,m}(L_h) = L_h^{-\widetilde{m}_\beta} p_\ell(L_h) p_r(L_h)^{-1}$, *where* m *indicates the degree of the polynomial in the numerator of the rational approximation. Under all the previous assumptions, we have that for every* $\beta > d/4$ *and every* $\varepsilon > 0$ *there exists* $m \in \mathbb{N}$ *such that*

$$\|L^{-2\beta} - r_{\beta,m}(L_h)\|_{L_2(L_2(\mathcal{D}))} \leq h^{\min\{4\beta - d/2 - \varepsilon, 2\}}.$$

Convergence rates for the covariance-based approximation

Theorem

Let $r_{\beta,m}(L_h) = L_h^{-\widetilde{m}_\beta} p_\ell(L_h) p_r(L_h)^{-1}$*, where* m *indicates the degree of the polynomial in the numerator of the rational approximation. Under all the previous assumptions, we have that for every* $\beta > d/4$ *and every* $\varepsilon > 0$ *there exists* $m \in \mathbb{N}$ *such that*

$$\|L^{-2\beta} - r_{\beta,m}(L_h)\|_{L_2(L_2(\mathcal{D}))} \leq h^{\min\{4\beta - d/2 - \varepsilon, 2\}}.$$

Idea: The idea is based on Cox and Kirchner's (2020) paper. There they obtained some bounds on operator's norm, and we use several times the inequality

$$\|AB\|_{HS} \leq \|A\| \|B\|_{HS}$$

and

$$\|AB\|_{HS} \leq \|A\|_{HS} \|B\|.$$

Where we show that the Hilbert-Schmidt norm appearing are bounded and the rate comes from the bounds in the operator's norm.

Consequences

- For an m-order approximation for the operator-based method, we can here choose the order as $2(m + m_\beta) - \widetilde{m}_\beta$ to have the same computational cost.
- This also enables us to represent the rational approximation $u_{h,m}^C \sim \mathsf{N}(0, L_h^{-\widetilde{m}_\beta} p_\ell(L_h) p_r(L_h)^{-1})$ as

$$u_{h,m}^C \overset{d}{=} \sum_{k=1}^{m} u_{r,k}, \qquad u_{r,k} \sim \mathsf{N}(0, a_k L_h^{-\widetilde{m}_\beta}(L_h + b_k I)^{-1})$$

where the $u_{r,k}$ are independent and $a_k, b_k > 0$.

Finite element representation

In terms of the finite element representation, the covariance matrix is given by

$$\mathbf{\Sigma}_{\hat{\mathbf{u}}} = (\mathbf{L}^{-1}\mathbf{C})^{\widetilde{m}_\beta} \sum_{k=1}^{m} a_k (\mathbf{L} + b_k \mathbf{C})^{-1} + \mathbf{K}$$

where:

$$\mathbf{K} = \begin{cases} k\mathbf{C} & \widetilde{m}_\beta = 0 \\ k\mathbf{L}^{-1}(\mathbf{C}\mathbf{L}^{-1})^{\widetilde{m}_\beta - 1} & \widetilde{m}_\beta >= 1 \end{cases}$$

and $\mathbf{L}$ is the matrix of the operator L_h in terms of the basis functions of V_h.

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- This was called the "Markov approximation" in the paper by Lindgren et al. (2011)
- In the numerical analysis literature this is known as the lumped mass method.

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- It is known in numerical analysis that in our finite element setup the replacement of the mass matrix **C** by the lumped mass matrix $\widetilde{C}$ is equivalent to replace the L_2 inner product by a quadrature approximation.
- We will then use some tools from numerical analysis to obtain our rate of convergence.

The lumped mass method

- Fix some $h \in (0,1)$ and an element $\tau \in \mathcal{T}_h$. Let $V_{1,\tau}, \ldots, V_{3,\tau}$ be the vertices of the triangle τ and consider the quadrature formula

$$Q_{\tau,h}(f) = \frac{area(\tau)}{3} \sum_{j=1}^{3} f(V_{j,\tau}).$$

- Obs: The quadrature scheme given above is exact for polynomials of degree less or equal to 1.
- Consider the following quadrature scheme to approximate the L^2-inner-product:

$$\langle f, g \rangle_h = \sum_{\tau \in \mathcal{T}_h} Q_{\tau,h}(fg).$$

The lumped mass method

Take $\{\varphi_i\}_{i=1}^{n_h}$ to be the set of standard basis of V_h consisting of the "hat" basis functions, which are continuous and piecewise linear, defined with respect to the triangulation $\mathcal{T}_h$ in such a way that if $\{V_i\}_{i=1}^{n_h}$ are the vertices of the triangulation $\mathcal{T}_h$, then $\varphi_i(V_j) = \delta_{ij}$. Then, we have that for every $j = 1, \ldots, n_h$

$$\|\varphi_j\|_h^2 = \langle \varphi_j, \varphi_j \rangle_h = \sum_{k=1}^{n_h} \langle \varphi_j, \varphi_k \rangle_{L^2(\mathcal{D})}$$

and that

$$\langle \varphi_i, \varphi_j \rangle_h = 0$$

if $i \neq j$ as $\varphi_i(x)\varphi_j(x)$ vanishes at all vertices of $\mathcal{T}_h$ (see Thomee, chapter 15 and also to Jin, Lazarov and Zhou (2013)) for further details.

- Define the operator $\widetilde{L}_h : \widetilde{V}_h \to \widetilde{V}_h$ as

$$\langle \widetilde{L}_h \phi_h, \psi_h \rangle_h = a_h(\phi_h, \psi_h) = \langle \boldsymbol{H} \nabla \phi_h, \nabla \psi_h \rangle_h + \langle \kappa^2 \phi_h, \psi_h \rangle_h .$$

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- One can also define the operator $\widetilde{L}_h : \widetilde{V}_h \to \widetilde{V}_h$ as

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- In this case we are only replacing the mass matrices "outside" of the operator.

Rate of convergence of the lumped mass operator

We have the following theorem:

Theorem

Under all the previous assumptions and additionally, if $\beta > 1/2$ and $\boldsymbol{H}$, κ belong to $W^{2,\infty}(\mathcal{D})$, then,

$$\|(L_h^{-2\beta} - \widetilde{L}_h^{-2\beta})\Pi_h\|_{L(H^1(\mathcal{D}))} \leq Ch^2.$$

Obs: If we consider the second form of discretization, the above rate hold under the usual assumptions on $\boldsymbol{H}$ and κ.

Rate of convergence of the lumped mass operator in the Hilbert-Schmidt norm

Theorem

Under all the previous assumptions and additionally, if $\beta > 1$ *and* $\boldsymbol{H}$, κ *belong to* $W^{2,\infty}(\mathcal{D})$, *then,*

$$\|(L_h^{-2\beta} - \widetilde{L}_h^{-2\beta})\Pi_h\|_{L_2(H^1(\mathcal{D}))} \leq Ch^2.$$

The idea of the proof is to use sharp rates of the quadrature approximation of the L_2-inner product with the strategy used for the covariance-based rational approximation.

Maximum likelihood estimation

We now move our attention towards maximum likelihood estimation of the parameters of the model.
To this end, consider the SPDE:

$$\tau(\kappa^2 I - \Delta)^{\beta} u = \mathcal{W},$$

where τ, κ and β are assumed to be positive constants.

Identifiability

We have the following theorem by Bolin and Kirchner (2020):

Theorem

Let $\mathcal{D} \subset \mathbb{R}^d$ *be bounded, open and connected. For* $i \in \{1, 2\}$*, let* $\beta_i > d/4$*,* $\kappa_i, \tau_i > 0$*, and consider the Gaussian measure* $\mu_i := N(m_i, \mathcal{Q}_i^{-1})$ *on* $L_2(\mathcal{D})$ *with mean* $m_i = 0$ *and precision operator* $\mathcal{Q}_i = \tau_i^2 L_i^{2\beta_i}$*, where for* $i \in \{1, 2\}$*, the operators* $L_i = \kappa_i^2 I - \Delta$ *are augmented with the same homogeneous Neumann or Dirichlet boundary conditions. Then,* μ_1 *and* μ_2 *are equivalent if, and only if,* $\beta_1 = \beta_2$ *and* $\tau_1 = \tau_2$*.*

Consistency of the MLE of τ

In the first scenario, assume that we have a sequence of finite sample points $\mathcal{D}_n$ whose union is dense in $\mathcal{D}$. Then,

Proposition

Assume $\beta > d/4$ is known, and let $\widehat{\tau}_n^2$ be the MLE of τ based on the sample $u(s_1), \ldots, u(s_{k_n})$, where $s_i \in \mathcal{D}_n$ and u is the solution of

$$\tau(\kappa^2 I - \Delta)^\beta u = \mathcal{W}.$$

Then, $\widehat{\tau}_n^2$ is weakly consistent.

Consistency of the MLE of τ

In the second scenario, we also assume we have a sequence of finite sample points $\mathcal{D}_n$ whose union is dense in $\mathcal{D}$. We consider the same equation, we also consider a finite element approximation of u, given by

$$u_{n_h}(\mathbf{s}) = \sum_{i=1}^{n_h} w_i \varphi_i(\mathbf{s}).$$

Let A be an $n \times n_h$ matrix with i, jth entry given by $\varphi_j(\mathbf{s}_i)$. Then, we assume we observe $\mathbf{y} = (y_1, \ldots, y_n)$, where

$$\mathbf{y} = A\mathbf{w}.$$

Proposition

Assume $\beta > d/4$ is known, and let $h := h(n)$ be any sequence such that $n \leq n_h$ for every n. Let $\widehat{\tau}_n^2$ be the MLE of τ based on the sample $y_1, \ldots, y_n$. Then, $\widehat{\tau}_n^2$ is weakly consistent.

Consistency of the MLE of τ

Idea: Combine the identifiability with an explicit expression of the maximum likelihood estimator and the law of large numbers.

Consistency of the MLE of τ

Finally, in the third scenario we replace, in the likelihood equation, the mass matrix by the lumped mass matrix. Then, we have the following Proposition:

Proposition

Consider the same assumptions of the second scenario. Additionally, assume $d = 2$ and $\beta > 1/2$ is known, and let $h = h(n)$ be chosen such that $nh^2 = o(1)$ and $n \leq n_h$ for every n. Let $\widehat{\tau}^2_{n,lump}$ be the MLE of τ based on the sample $y_1, \ldots, y_n$ with the mass matrix replaced by the lumped mass matrix. Then, $\widehat{\tau}^2_{n,lump}$ is weakly consistent.

Idea of the proof: Use the bound in operator's norm for the lumped mass covariance operator to bound the difference $|\widehat{\tau}^2_{n,lump} - \widehat{\tau}^2_n|$.

Thank you!