Dyson models with random boundary conditions: limit behaviour and metastates

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Long-range Ising models, in $d = 1$, (Dyson models),
long-range,
ferromagnetic Ising models with pair interactions.
Ising spins: random variables $\omega_i = \pm 1, i \in Z^d$.
Formally long-range Hamiltonian.
Can be done in different dimensions,
here concentrate on $d = 1$.
$H = \sum_{i,j \in Z^d} J(i-j)\omega_i \omega_j$.
with polynomial decay, e.g. $J(i-j) = -|i-j|^{-d\alpha}$.
Simulates high dimensions.
Varying decay power α between 1 and 2 is like
varying dimension in short-range models,
but possible in continuous way.
Slower decay corresponds to higher dimension.

Phase transitions possible, even in $d = 1$.
Multiple Gibbs measures.
(Dyson (1969), proving Kac-Thomson conjecture).
Different proofs since.
Use approach of Cassandro-Ferrari-Merola-Presutti,
(plus Littin-Picco, plus theses of Littin and Kimura).
Buzzwords: "Contours, low-temperature expansion".

1) Approximately it holds
$\alpha \approx \frac{d+2}{d}$ for critical behaviour,
mean-field critical behaviour
for $\alpha < \frac{3}{2}$, like $d > 4$.
Suggestive, but only approximate guide.
2) For surface-to-volume arguments $\alpha \approx \frac{d+1}{d}$.
3) Never, for no α
rigid interfaces in $d = 1$.
As in $d = 2$.
4) Here I argue:
For $\alpha > \frac{3}{2}$ random boundary conditions
provide finite boundary energy.
As in $d = 1$.
For $\alpha < \frac{3}{2}$, boundary energy diverges.
As in higher d.

Random boundary conditions.
Boundary conditions outside a volume
independent of interactions.
Relevant especially for "quenched" disordered systems,
such as spin glasses.
Exact relation for "Mattis disorder":
Let $J_{i,j}$ be ferromagnetic.
Site disorder $\eta_i = \pm 1$,
then Mattis spin-glass interaction
$J'_{i,j}(\eta) = J_{i,j} \eta_i \eta_j$.
Mattis spin glass with fixed boundary conditions
equals ferromagnet with random boundary conditions.
Equivalence via
random gauge transformation
$\sigma'_i = \eta_i \sigma_i$.

Energy estimates:
Flipping all spins in interval of length L
costs energy, boundary term,
maximally $O(L^{2-\alpha})$,
uniformly bounded energy when $\alpha > 2$.
Maximal energy between two half-lines is bounded.
Main ingredient for Gibbs state uniqueness
(and analyticity, etc). Long known.
New question:
Maximal energy from **random** boundary conditions?
Influence on limit behaviour of Gibbs measures?
Dependence on α?

Gibbs (=DLR) measures= Gibbs fields=
" almost" Markov random fields.
Discovered independently,
in East (mathematics)
and West (physics),
(Dobrushin, Lanford-Ruelle 60's).
Mathematical Physics.
Here two-state -Bernoulli- variables,
(= **Ising** spins:)
$\omega_i = \pm 1$, for all $i \in Z$.
Warning: DLR Gibbs $\neq$ SRB Gibbs.

Gibbs measures:
Let G be an infinite graph, here Z.
Configuration space:
Space of sequences: $\Omega = \{-,+\}^G$.
Probability measures on Ω,
labeled by **interactions**.
An interaction is a collection of functions,
$\Phi_X(\omega)$, dependent on $\{-,+\}^X$,
where the X are subsets of G.

Energy (**Hamiltonian**)
$H_\Lambda^{\Phi,\tau}(\omega) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
Sum of interaction-energy terms.
A measure μ is *Gibbs* iff:
(A version of) the
conditional probabilities of
finite-volume configurations,
given the outside configuration, satisfies:
$\mu(\omega_\Lambda | \tau_{\Lambda^c}) = \frac{1}{Z_\Lambda^\tau} \exp -\beta \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
for ALL
configurations ω,
boundary conditions τ
and finite volumes Λ,
at inverse temperature β.

Gibbsian form.
Rigorous version of
"$\mu = \frac{1}{Z} \exp -\beta H$",
Gibbs canonical ensemble.
Larger energy means
exponentially smaller probability.
All extremal Gibbs measures
obtainable as infinite-volume limits
with suitable boundary conditions.
Non-extremal Gibbs measures sometimes (Coquille).

In Dyson models
at low temperature (largeβ):
Two different extremal Gibbs measures,
for the same interaction,
called μ^+ and μ^-, for such Φ.
Plus convex combinations:
$\mu_\lambda = \lambda\mu^+ + (1-\lambda)\mu^-$.

Random boundary conditions and metastates.

Ferromagnets with random boundary conditions
act to some degree as toy examples
of quenched disordered systems.
Hamiltonians and Gibbs measures disorder-dependent
(dependent on disorder random variables η).
Here disordered boundary conditions.
They can display non-convergence
of the sequence of finite-volume measures
in the thermodynamic limit
(Chaotic Size Dependence).

Instead:
Convergence in distribution
to objects called
“metastates.”
Random distributions on Gibbs measures,
which are (a subset of) the possible limit points
(Newman-Stein, Aizenman-Wehr).
Concepts developed for Spin Glasses.
Most complicated disordered spin systems.
Random boundary conditions ferromagnets are
among the simplest disordered spin systems.
Physically they are like
fixed boundary conditions for spin-glasses.
Rigorously true for Mattis disorder.

In general this distribution on Gibbs measures
is random (η-dependent) object.
(Metastate is measure on measures on measures).
Translation covariance, needs proof.
Here not needed.
Simplifications for us:
1) the Gibbs measures are independent of the disorder,
2) they are translation invariant.
Still we obtain a proper distribution:
The metastate is “dispersed”,
and has thus a support consisting of
more than one Gibbs measure.

WARNING:

Metastates are different from **mixtures**.

Mixtures are **measures** on spins.

Metastates are

(measures on) measures on measures on spins.

Example 1:

Periodic boundary conditions produce

a **non-dispersed** metastate

on mixed symmetric Gibbs measure (n.n. 2d Ising).

$\mu_{symm} = \frac{1}{2}(\mu^+ + \mu^-)$ and $\kappa = \delta_{\mu_{symm}})$.

Example 2:

Random boundary conditions produce

a **dispersed** metastate

on pure plus and minus states (n.n. 2d Ising).

$\kappa = \frac{1}{2}(\delta_{\mu^+} + \delta_{\mu^-})$.

Consider a sufficiently sparse increasing sequence of intervals $\{-L_n, +L_n\}$ for Dyson models with random (Bernoulli) boundary conditions η, with $\eta_i = \pm$.

Question:

What could the limit points be of the sequence of finite-volume measures $\mu^{\alpha,\eta}_{L_N}$, when N diverges?

Answer:

Depends on α.

Case 1)

Interaction across the boundary diverges when $\alpha < \frac{3}{2}$.

Like higher-dimensional short-range models.

Case 2)

Boundary energy remains bounded otherwise.

Proof idea:
Let $W = \sum_{i<0,j>0} \eta_i |i-j|^{-\alpha} = \sum_{i<0} \eta_i |i|^{1-\alpha}$,
the interaction energy between half-lines,
the plus configuration
on the positive half-line (Dyson ground state)
and a random configuration (boundary condition)
η on the negative half-line.
Then $EW = 0$, and
$EW^2 = \sum_{i<0} |i|^{2-2\alpha}$,
finite for $\alpha > \frac{3}{2}$,
infinite otherwise.

Remark on **energy estimates**:

1) Maximal energy of interval of length L with plus boundary conditions is $L^{2-\alpha}$.

Explains phase transition.

2) Maximal energy of interval of length L with random boundary conditions is $L^{\frac{3}{2}-\alpha}$.

Explains metastate behaviour at low T.

(Similar to a one-dimensional long-range spin-glass ground state estimate).

3) Energy (square) expectation of two random configurations on infinite half-lines left and right

$\sum_{i<0,j>0} |i-j|^{-2\alpha}$ is finite for all $\alpha > 1$.

High T ferromagnet.

(Similar to positive-T long-range spin-glass (free-energy) estimates).

Remark:
An interval of length N interacting with a half-line, for $1 < \alpha < \frac{3}{2}$ has energy $O(N^{\frac{3}{2}-\alpha})$.
This holds both
for a plus interval interacting with a random half-line;
and for a random interval interacting with a plus half-line.

Consequences:

Case 1):

When $\alpha < \frac{3}{2}$,
the limit points are determined by the
signs of the diverging boundary term.
Thus one obtains
either the plus Gibbs measure μ^+
or the minus Gibbs measure μ^-.
The dispersed metastate Γ
is the average of those two:
$\Gamma = \frac{1}{2}(\delta_{\mu^+} + \delta_{\mu^-})$.
Like higher-dimensional, $d > 1$, short-range Ising models.
Boundary energy shows Gaussian, central limit, behaviour
and -more precisely- a weak form of a local limit theorem.

Case 2):
When $\alpha > \frac{3}{2}$: New behaviour
The boundary energies converge
to some well-defined random variable.
Thus the sequence of finite-volume measures $\mu^{\alpha,\eta}_{L_N}$
now has as limit points Gibbs measures which are
mixtures of the plus and minus measures:
$\mu_\lambda = \lambda\mu^+ + (1-\lambda)\mu^-$.
As a consequence the dispersed metastate becomes
an average over these mixtures:
$\Gamma = \int P_\alpha(d\lambda)\delta_{\mu_\lambda}$,
with the measure on the mixtures P_α
possibly dependent on the details of the model,
like the value of α.

Remark:
A metastate construction with support on extremal Gibbs measures can always be made from metastates with mixed measures in its support (Cotar, Külske, Jahnel).
Of course in examples
one can usually also do this by hand.
Here also.

Technically hardest part is controlling the
diverging boundary term.
It cannot be too large
(proof by exponential Chebyshev).
It cannot be too small
(proof by weak form of local limit theorem.)
We can write
$W_N^{\eta}(\sigma) = \sum_{i<0, j=1...N} |i-j|^{\alpha} \eta_i \sigma_j = \sum_{j=1..N} h_j(\eta)\sigma_j$,
with correlated random magnetic fields h_j.
$P(|h_j| > |j|^{\frac{1}{2}+\varepsilon}) < \exp -|j|^{\varepsilon}$,
so Borel-Cantelli plus union bound
tell us that after an initial interval ALL h_j
almost surely will have satisfy this decaying upper bound.
This translates in the W_N being not too large,
so the low-temperature contour analysis survives.

In the other direction, not too small:
We want to control the characteristic functions
$\Phi_N(t) = E \exp it \ln Z(H_N + W_N(\eta))$,
again by an expansion.
Convergence in t improves in N, away from the boundary,
distance from boundary also increasing with N.
We obtain that the probability
$P(W_N < N^{\frac{3}{2}-\alpha-\varepsilon}) < N^{-\varepsilon}$,
that is, the probability that
we are in a window smaller than the CLT scaling,
goes to zero.
The rest is Borel-Cantelli.

Conclusions:
Long-range Ising models often behave like
higher-dimensional short-range models.
But this holds in **some** but **not all** respects.
Similar behaviour of metastability.
Nucleation via critical droplets.
Similar behaviour in models in inhomogeneous fields.
Imry-Ma surface-volume arguments.
Different behaviour of interfaces.
Never interfaces for Dyson models.
Different behaviour of metastates.
Dyson models can have metastates on mixed Gibbs measures.
With E. Endo and A. Le Ny,
in progress.