Contact process under renewal cures

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with

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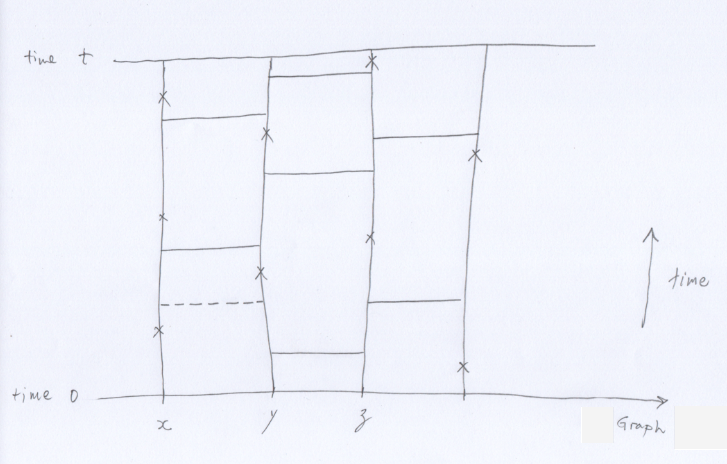
(SPA, Bernoulli, arXiv)

Model for spread of an infection (Harris 1974)



$\times$: cure times; T, $T_{x,i}$ iid; —: infection times; PPP(λ)

Question

P(infection started at the origin survives forever) > 0 *for all* $\lambda > 0$?

Remark

Not true for the usual contact process ($T \sim \mathrm{Exp}(1)$):
$\exists\ \lambda_c > 0$ such that P(survival) $= 0$ for $\lambda < \lambda_c$

Proof

Comparison to branching

Sufficiency for P(survival) > 0 for all $\lambda > 0$ on $\mathbb{Z}^d$

Theorem 1*

We have survival wpp for all $\lambda > 0$ if $P(T > t) > t^{-\alpha}$ for all large t, for some $\alpha < 1^\dagger$, plus regularity conditions.

Example

T attracted to α-stable law, $0 < \alpha < 1$.

Extension‡

Same for $P(T > t) = \frac{L(t)}{t}$, L slowly varying and $\nearrow$ fast at ∞ †.

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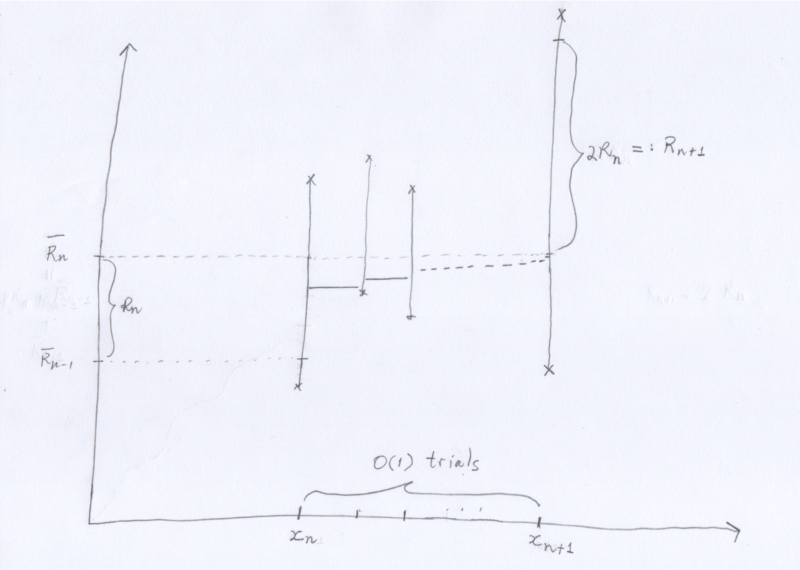
*Marchetti, Mountford, Vares, F (SPA '18)

†In particular $E(T) = \infty$.

‡Mountford, Ungaretti, Vares, F (arXiv)

A key ingredient of proof: tunneling event $\mathcal{T}_n$



$\prod_{n\geq 1} P(\mathcal{T}_n) > 0$ for all $\lambda > 0$

Extensions/refinements: complete convergence

Theorem 2[§]

Let the distribution of T satisfy the conditions of Theorem 1. Then, for a RCP starting from any initial condition $\xi_0 \in \{0,1\}^{\mathbb{Z}^d}$ we have that ξ_t converges in law, as $t \to \infty$, to

$$P(\tau < \infty)\delta_{\underline{0}} + P(\tau = \infty)\delta_{\underline{1}},$$

where $\tau = \inf\{t > 0 : \xi_t \equiv 0\}$ is the *extinction time*.

[§] Mountford, Vares, Ungaretti, F (arXiv)

Extensions/refinements: closeness to determinism

Let $\mathcal{G}$ denote the σ-field generated by the renewal processes and the extinction random time τ.

Theorem 3[¶]

If T is attracted to an α-stable law, $0 < \alpha < 1$, then for all $x \in \mathbb{Z}^d$:

(i) If $\alpha < 1/2 +$ reg'ty cond, it holds on $\{\tau = \infty\}$ that

$$\lim_{t\to\infty} \left| P(\xi_t(x) = 0 \mid \mathcal{G}) - e^{-2d\lambda Y_t(x)} \right| = 0 \text{ a.s.};$$

(ii) If $\alpha > 1/2$ and $F(t) > 0 \ \forall\ t > 0$, it holds on $\{\tau = \infty\}$ that

$$\overline{\lim_{t\to\infty}} \left| P(\xi_t(x) = 0 \mid \mathcal{G}) - e^{-2d\lambda Y_t(x)} \right| > 0 \text{ a.s.},$$

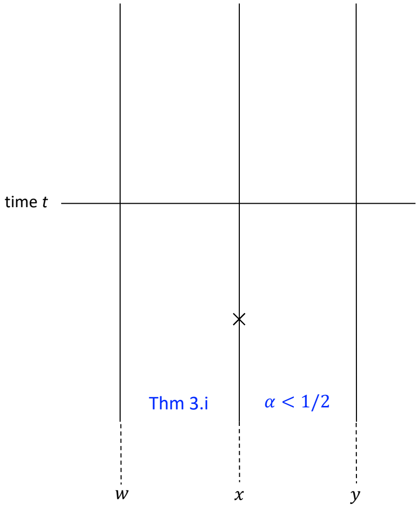
where $Y_t(x)$ is the age of the renewal process in x at time t.

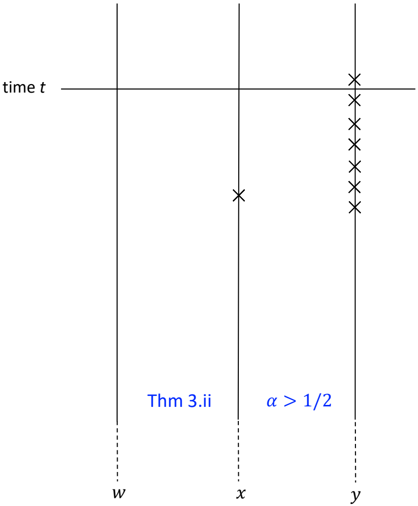
Refinement[¶]

Precise asymptotics of max in t of $P(\xi_t(x) = 0 \mid \mathcal{G})$ as fn of $Y_t(x)$.

[¶] Mountford, Vares, Ungaretti, F (arXiv)

Thm 3: what is going on





(conditioning on *all* cure marks and survival)

Sufficiency for "P(survival) $= 0$ for some $\lambda > 0$" on $\mathbb{Z}^d$

Theorem 4[‖]

If $E(T^2) < \infty$, then $P(\text{survival}) = 0$ for small $\lambda > 0$.

Proof: Standard supermartingale argument/comparison to branching

‖ Mountford, Vares, F (SPA '20)

Tricky case

Remark 1

Thm's 1 and 4 leave a gap: T having a 1st but not a 2nd moment; then comp to branching/supermart argument does *not* hold.

Remark 2

Tunneling argument does not work either.

Theorem 5**

If $E(Te^{\theta\sqrt{\log T}}) < \infty$ for some const θ large enough, then

$$P(\text{survival}) = 0 \text{ for small } \lambda > 0.$$

Remark 3

Mountford, Vares, F (SPA '20) have much stricter conds for this:

$$E(T^\alpha) < \infty \text{ for some } \alpha > 1;\ d = 1;\ T \text{ cont's and } \frac{f(t)}{1-F(t)} \searrow$$

Extensions and generalizations

Hilário, Ungaretti, Valesin, Vares; arXiv:2108.03219

**Mountford, Vares, Ungaretti, F (arXiv)

Finite graphs

Consider the RCP on a finite connected graph $G = (V, E)$ with distr of T attracted to an α-stable law, $\frac{1}{2} < \alpha < 1$.

Theorem 6[††]

For all $\lambda > 0$

1. $P(\text{survival}) = 0, \quad \text{if} \quad |V| < v^- := 2 + \frac{2\alpha-1}{(1-\alpha)(2-\alpha)}$;
2. $P(\text{survival}) > 0, \quad \text{if} \quad |V| > v^+ := \frac{1}{1-\alpha}$.

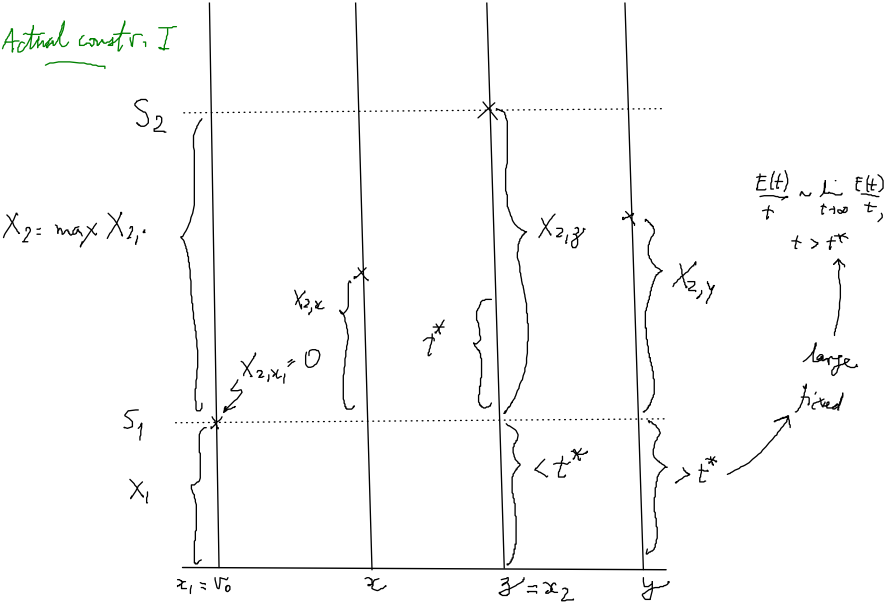
Remarks

1) $\alpha \leq \frac{1}{2}$: 1 holds trivially; for 2, need usual extra reg'ty on distr of T (in order to be able to apply SRT);

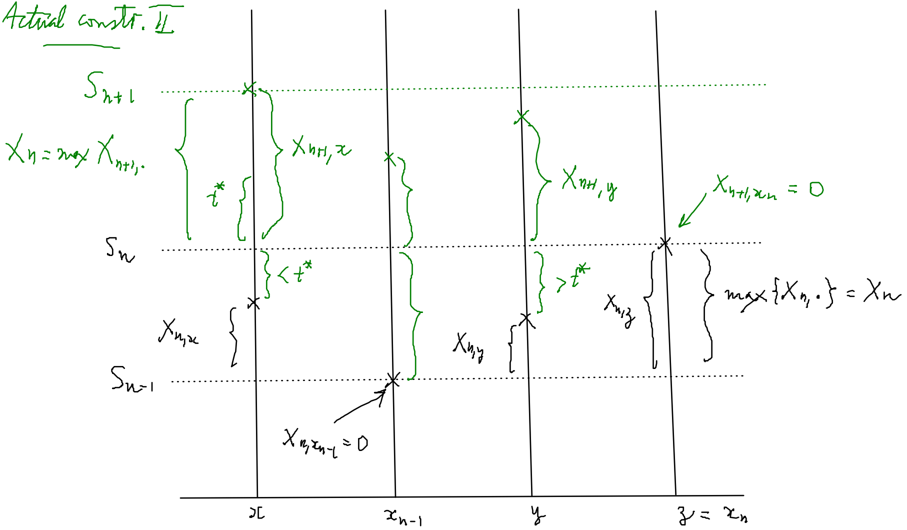
2) $v^+ - v^- < 1$ for all $\alpha \in (\frac{1}{2}, 1)$: if $[v^-, v^+] \cap \mathbb{Z} = \emptyset$, then above criteria determine situation for all V; otherwise, the situation is undetermined for exactly one value of $|V|$.

[††] Gomes, Sanchis, F (Bernoulli '21)

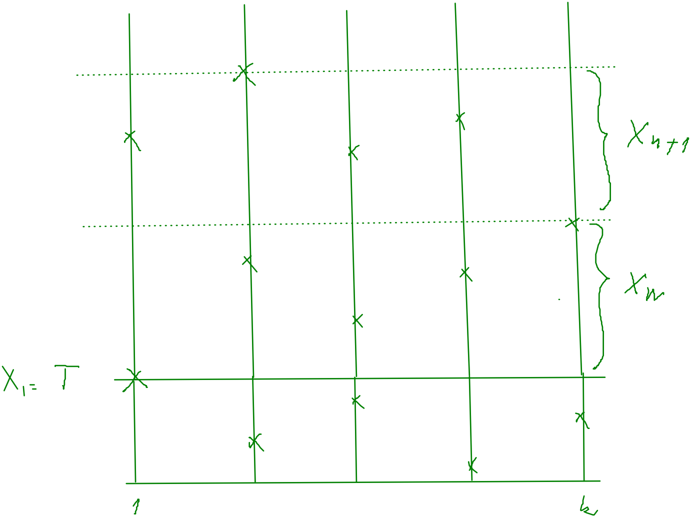
Rough, approx ideas of proof — Extinction



Actual constr. II



Approx. constr. & argument (a bit misleading)



$$X_{n+1} \overset{d}{\leq} X_n \max\{\underbrace{Y_1^n, \dots, Y_{k-1}^n}_{\text{iid}}\}, \qquad f_{Y_1^n}(t) = \text{const} \frac{1}{t^\alpha (1+t)}, \; t > 0$$

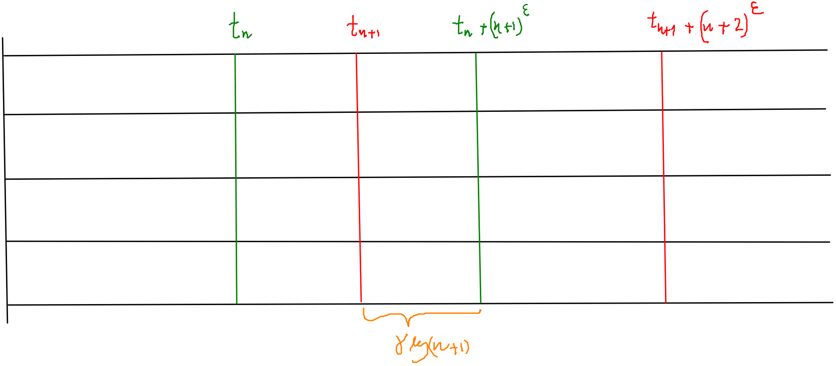
$$Z_n = \frac{X_{n+1}}{X_n} \overset{d}{\leq} \max\{Y_1^n, \dots, Y_{k-1}^n\}$$

$$X_{n+1} \overset{d}{\leq} Z_n \times Z_{n-1} \times \dots \times Z_1 \quad \Rightarrow \quad \log X_{n+1} \overset{d}{\leq} \sum_{i=1}^{n} \log Z_i$$

$$E \log Z = \int_0^\infty \log z \, f_Z(z) \, dz < 0$$

Survival

$$t_n = \sum_{j=n_0}^{n} \left(j^{\varepsilon} - \gamma \log j \right) \sim n^{1+\varepsilon}$$



Idea: 1) $\forall\, n$ large, at least one x w. no cure in $(t_n, t_n + (n+1)^{\varepsilon})$

2) - - - - transmission of infection on at least one time subint.

3) initial transm. w. > 0 prob.

1)

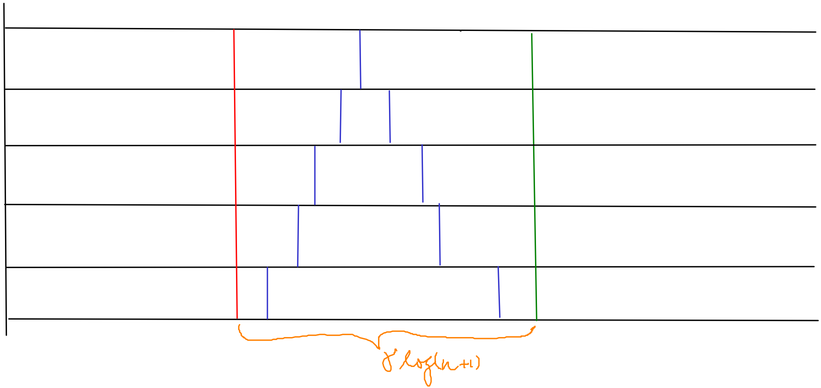
$$A_n = \left\{ \max_{x \in V} E_x(t_n) > n^{\varepsilon} \right\}$$

$$\to P(A_n^c) \leq n^{-\beta}, \quad \beta = \underbrace{|V|}_{k} (1 - \alpha - 3\varepsilon) > 1$$

$$P(A_n^c) = P\left(\frac{E_x(t_n)}{t_n} \leq n^{-1} \right)^k \approx \left(\int_0^{n^{-1}} x^{-\alpha} dx \right)^k \sim n^{-k(1-\alpha)}$$

BC: $\underline{A_n}$ occurs $\forall$ large n

2)



$e_1, \dots, e_\ell$: "spanning cycle"

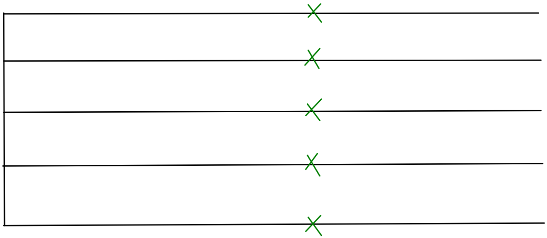
γ large

$$P(\underbrace{\exists \text{ stairway along cycle}}_{B_n}) \geq \left(1 - e^{-\frac{\lambda}{\ell}\gamma \log(n+1)}\right)^{\ell} = \left(1 - \frac{1}{(n+1)^{\delta}}\right)^{\ell}, \quad \delta = \frac{\lambda\gamma}{\ell} > 1$$

$$\sim 1 - \frac{\ell}{n^{\delta}}, \quad n \text{ large}$$

BC: B_n occurs $\forall$ large n

3)



P(eventual tie) < 1 : a.s. recurrence contradicts 1) !