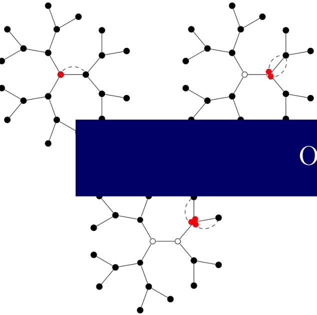
Probability Webinar / IM - UFRJ

November 8th, 2021



On some branching processes with selection

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About this talk!

- The accessibility percolation model on trees

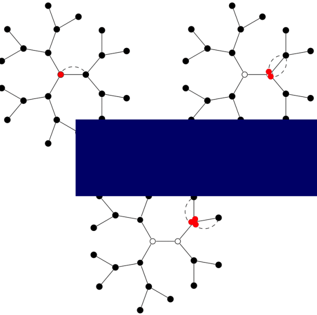
  Based on joint works with:

  - Cristian Coletti (UFABC), Renato Gava (UFSCar);
  - Daniela Bertacchi (Milano-Bicocca), Fabio Zucca (Poli. Milano).

- A branching random walk with barriers

  Based on a joint work with:

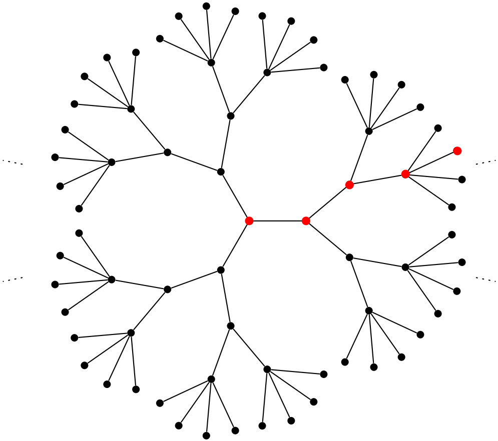
  - Cristian Coletti (UFABC), Nevena Marić (Union University).



Accessibility percolation

The problem

Ingredients: a rooted tree $T = (V, E)$ and $(X_v)_{v \in V}$ i.i.d. with $X_v \sim U(0,1)$



Interest: The existence of a path $v_1, v_2, v_3, \ldots$ such that $X_{v_1} < X_{v_2} < X_{v_3} < \cdots$

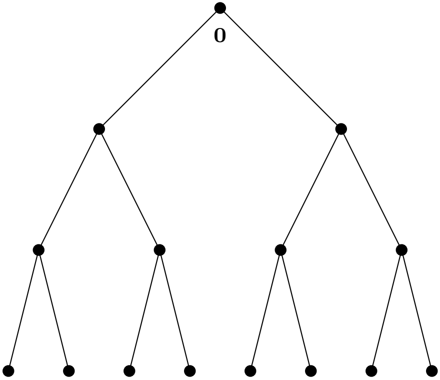
Motivation

This is a new percolation model inspired by evolutionary biology:

"Imagine a population of some lifeform endowed with the same genetic type (genotype). If a mutation occurs, a new genotype is created which can die out or replace the old one. Provided natural selection is sufficiently strong, the latter only happens if the new genotype has larger fitness. As a consequence, on longer timescales the genotype of the population takes a path through the space of genotypes along which the fitness is monotonically increasing." (Nowak and Krug, 2013)

The first work: n-ary trees

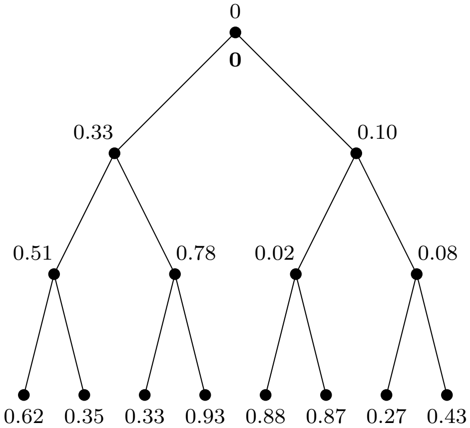
Nowak and Krug (Europhys. Lett. 2013)



Consider an n-ary tree with height h. Here: $n = 2$ e $h = 3$.

The first work: n-ary trees

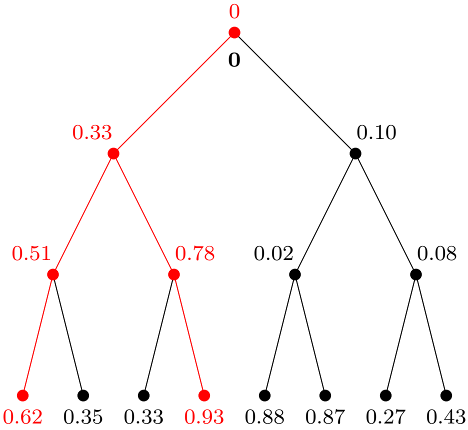
Nowak and Krug (Europhys. Lett. 2013)



For each vertex v we associate $X_v \sim Uniforme(0,1)$ from an i.i.d. collection

The first work: n-ary trees

Nowak and Krug (Europhys. Lett. 2013)



We count the number of paths $v_1, v_2, \ldots$ such that $X_{v_1} < X_{v_2} < \cdots$

The first work: n-ary trees

Let N_h be the number of such paths connecting $\mathbf{0}$ with the last level.

Theorem (Nowak and Krug, 2013)

If $n := n(h) = \alpha h$, *with* $\alpha > 0$ *is constant, then*

$$\lim_{h\to\infty} \mathbb{P}\left(N_h \geq 1\right) \begin{cases} = 0, & \text{if } \alpha \leq e^{-1}, \\ > 0, & \text{if } \alpha > 1. \end{cases}$$

Ref.: Nowak and Krug, *Accessibility percolation on n-trees*, Europhys. Lett. **101** (2013), 66004.

Idea of the proof of Nowak and Krug

$$\mathbb{E}(N_h) \geq \mathbb{P}\left(N_h \geq 1\right) \geq \frac{\mathbb{E}\left(N_h\right)^2}{\mathbb{E}\left(N_h^2\right)},$$

together with

$$\mathbb{E}\left(N_h\right) = \frac{n^h}{h!},$$

and

$$\mathbb{E}\left(N_h^2\right) \leq \mathbb{E}\left(N_h\right) + \mathbb{E}\left(N_h\right)^2 + \sum_{k=1}^{h-1} \binom{2k}{k} \frac{n^{h+k}}{(h+k)!}.$$

A natural question!

$$\ldots \textit{ what about } \alpha \in (e^{-1}, 1]\textit{?}$$

The answer

Theorem (Roberts and Zhao, 2013)

If $n = \alpha h$, with $\alpha > e^{-1}$, then

$$\lim_{h\to\infty} \mathbb{P}\left(N_h \geq 1\right) = 1.$$

Ref.: Roberts and Zhao, *Increasing paths in regular trees*, Electron. Commun. Probab. **18** (2013), 1-10.

If $n = \alpha h$ then $\alpha_c = e^{-1}$:

$$\lim_{h\to\infty} \mathbb{P}\left(N_h \geq 1\right) = \begin{cases} 0, & \text{if } \alpha \leq e^{-1}, \\ 1, & \text{if } \alpha > e^{-1}. \end{cases}$$

For a review of recents results . . .

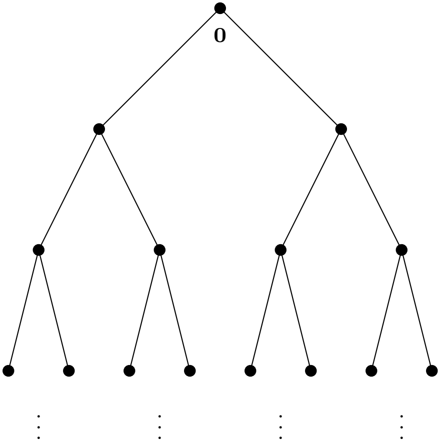
Accessibility percolation in random fitness landscapes,

by Joachim Krug (Institute for Theoretical Physics, University of Cologne).

To appear in “Probabilistic Structures in Evolution”.

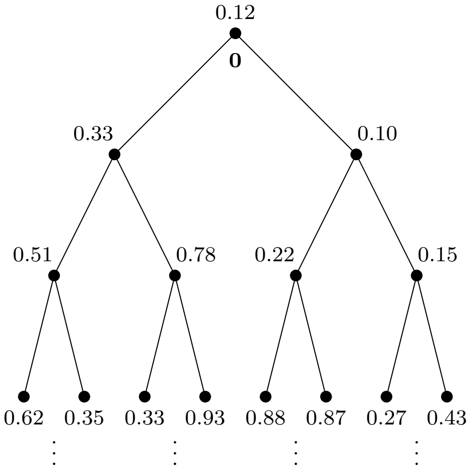
Preprint: arXiv:1903.11913 (68 references!)

The model on infinite trees



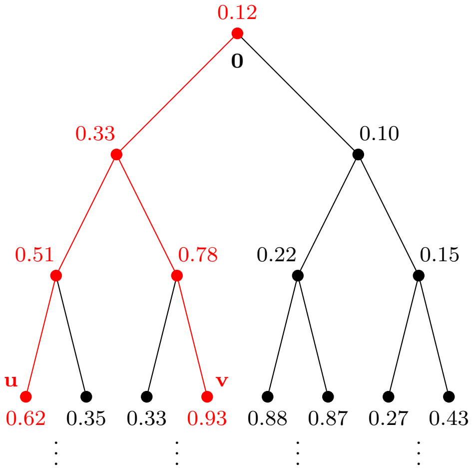
Consider an infinite, locally finite, rooted tree $T = (\mathcal{V}, \mathcal{E})$.

The model on infinite trees



Associate to each $v \in \mathcal{V}$ a r.v. $X_v \sim U(0,1)$ from an i.i.d. collection

The model on infinite trees



Is there an infinite path $v_1, v_2, \ldots$ such that $X_{v_1} < X_{v_2} < \cdots$ w.p. > 0?

Accessible paths

A path $v_0, v_1, \dots, v_n$ in T is said to be an accessible path if

$$X_{v_0} < X_{v_1} < X_{v_2} < \cdots < X_{v_n}.$$

We denote $v_0 \xrightarrow{a.p.} v_n$.

For each $n \geq 1$, let

$$\Lambda_n := \Lambda_n(T) = \{\mathbf{0} \xrightarrow{a.p.} v, \text{ for some } v \in \partial T_n\}.$$

Accessibility percolation

We say that there is accessibility percolation if the event

$$\bigcap_{n=1}^{\infty} \Lambda_n$$

occurs. We denote the percolation probability as

$$\theta(T) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} \Lambda_n\right) = \lim_{n\to\infty} \mathbb{P}(\Lambda_n).$$

Spherically symmetric trees

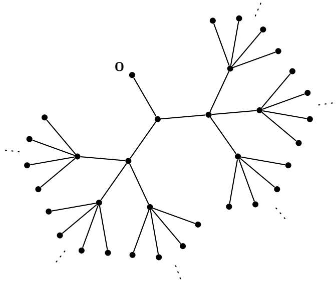
We say that T is a spherically symmetric trees with a growth function

$$f : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$$

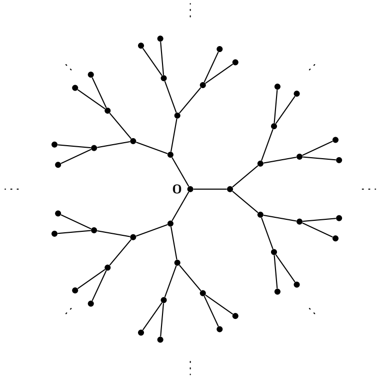
if:

- $degree(\mathbf{0}) = f(0)$, and
- $degree(v) = f(\mathrm{dist}(\mathbf{0}, v)) + 1$, for each $v \in V \setminus \{\mathbf{0}\}$.

Spherically symmetric trees



(a) Factorial tree $\mathbb{T}_!$.



(b) Homogeneous tree $\mathbb{T}_d$.

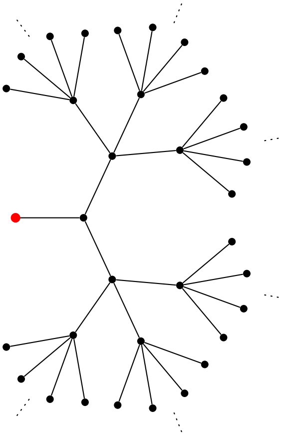
Question

What are the conditions on f ...

... to guarantee accessibility percolation w.p. > 0?

Example: Factorial tree

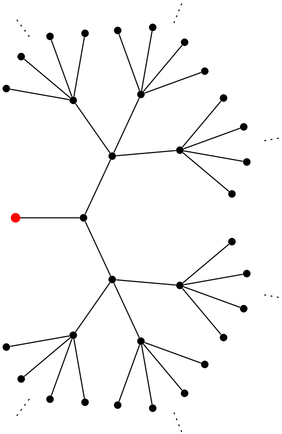
For the factorial tree $T_!$: $f(i) = i + 1$, $i \geq 0$



Example: Factorial tree

$|\partial T_{!,n}| = n!$

For the factorial tree $T_!$: $f(i) = i + 1$, $i \geq 0$



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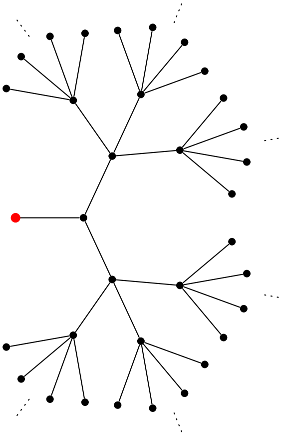

$$\mathbb{P}(\Lambda_n) = \mathbb{P}(N_n \geq 1) \leq \frac{|\partial T_{!,n}|}{(n+1)!} = \frac{1}{n+1}$$

Example: Factorial tree

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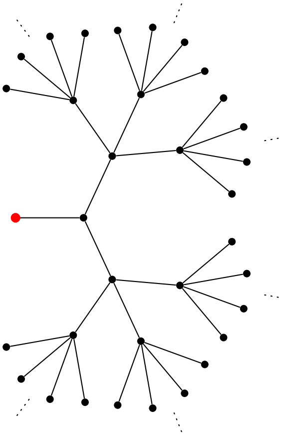


$$\mathbb{P}(\Lambda_n) = \mathbb{P}(N_n \geq 1) \leq \frac{|\partial T_{!,n}|}{(n+1)!} = \frac{1}{n+1}$$

and letting $n \to \infty$, we have $\theta(T_!) = 0$.

Phase transition

Proposition 1

Let T_α be an spherically symmetric tree with

$$f(i) = \lceil (i+1)^\alpha \rceil, \;\; i \geq 0$$

where $\alpha > 0$ is a constant. If $\alpha \leq 1$, then $\theta(T_\alpha) = 0$.

The proof is by coupling with $T_!$, since $T_\alpha \prec T_!$ and $\theta(T_!) = 0$.

Phase transition

Proposition 2

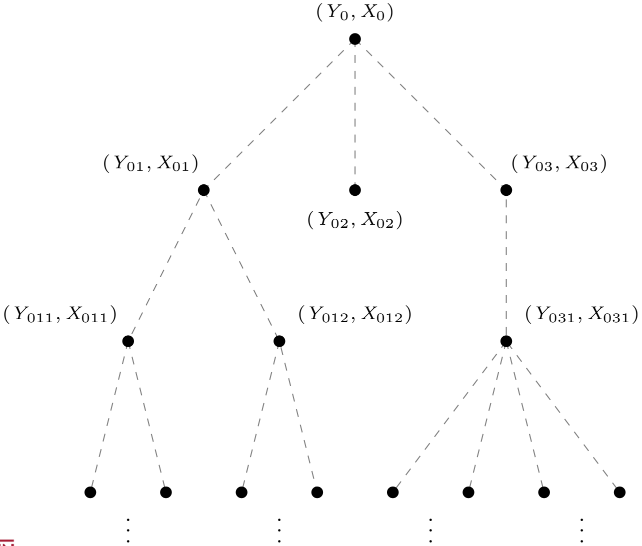
Let T_α be an spherically symmetric tree with

$$f(i) = \lceil (i+1)^\alpha \rceil, \quad i \geq 0$$

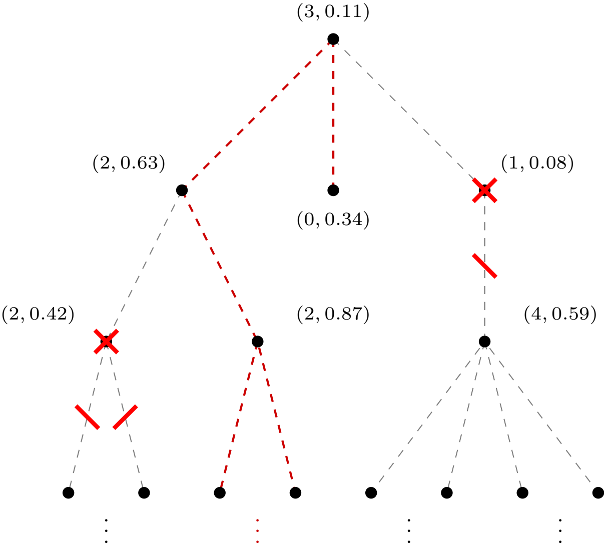
where $\alpha > 0$ is a constant. If $\alpha > 1$, then $\theta(T_\alpha) > 0$.

The proof is by coupling with a suitable defined branching process in varying environment, whose survival implies accessibility percolation in T_α.

Accessibility percolation and branching with selection



Accessibility percolation and branching with selection



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Survival for the BP with selection

Theorem (Bertacchi, R. and Zucca, 2020)

Assume that

$$\sum_{i=0}^{\infty} \frac{1}{m_i} < \infty,$$

and for some $C > 0$ *there exist* $g : \mathbb{N} \to [1, \infty)$ *such that*

$$\frac{m_n^{(2)}}{m_n^2} \leq g(n),$$

for n *sufficient large with*

$$\limsup_{n \to \infty} \left(g(n+1)/g(n) \right) < C.$$

Then, the BPS survives with positive probability.

For the case of the spherically symmetric trees ...

If T is an spherically symmetric tree with growth function f, then

$$\sum_{i=0}^{\infty} \frac{1}{f(i)} < \infty$$

implies accessibility percolation with positive probability.

Let T_α be an spherically symmetric tree with

$$f(i) = \lceil (i+1)^\alpha \rceil, \quad i \geq 0$$

where $\alpha > 0$ is a constant. Then,

$$\theta(T_\alpha) \begin{cases} = & 0, \quad \text{if } \alpha \leq 1, \\ > & 0, \quad \text{if } \alpha > 1. \end{cases}$$

Remains the question

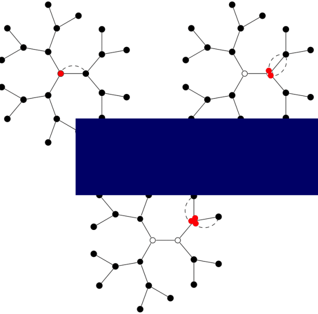
Can we find a necessary and sufficient condition on f ...

... to guarantee accessibility percolation with positive probability?

Our references for this model

On the existence of accessibility in a tree-indexed percolation model, joint work with C. Coletti (UFABC) and R. Gava (UFSCar), Physica A **492** (2018):382-388.

G-W processes in varying environment and accessibility percolation, joint work with Bertacchi (Milano-Bicocca) and Zucca (Politecnico Milano), Brazilian Journal of Probability and Statistics **34** (2020):613-628.

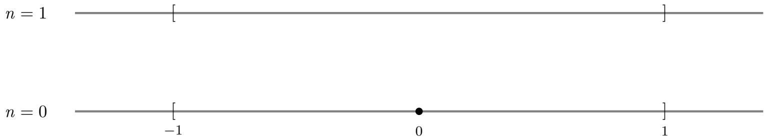


A branching random walk with barriers

The problem

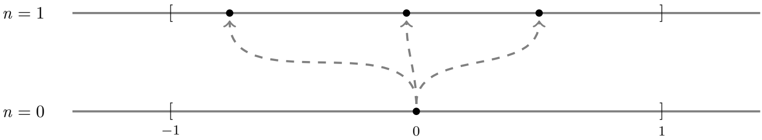
Initially there is a one particle located at the origin.

The problem



Its descendants are scattered in $[-L, L]$ according to a $P.P.P.(\lambda/2)$.

The problem



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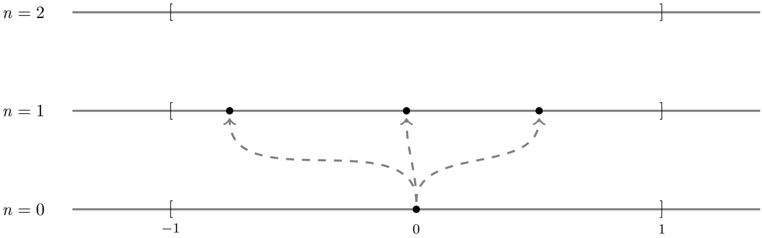
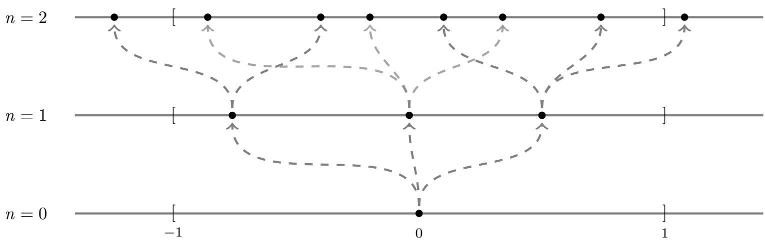


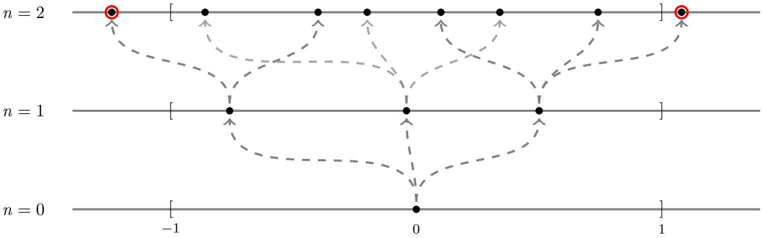
A particle located at x try to give rise to particles scattered at $[x - 1, x + 1]$.

The problem



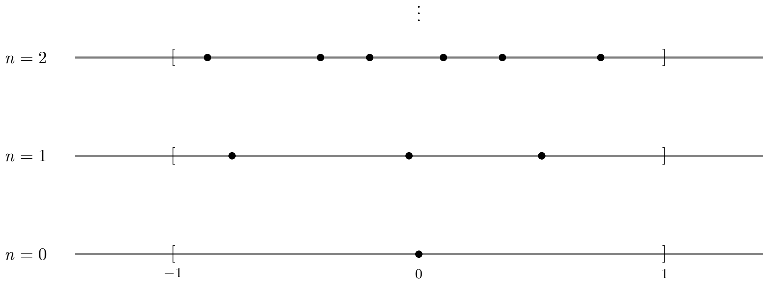
This is done according to a $P.P.P.(\lambda/2)$.

The problem



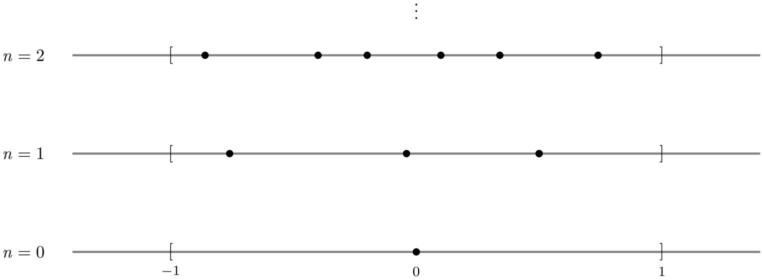
But only attempts inside $[-L, L]$ are considered successful.

The problem



Each generation is given by successful births and their parents are dead.

The problem



Our interest is in the survival or extinction of this process.

Motivation

This is a mathematical model inspired in the phenomenon of survival for plant species in a reduced habitat.

The process and its survival

For $n \geq 0$ let $\mathcal{Y}_n$ be the set of particles alive in generation n, and let

$$\mathcal{Y} := (\mathcal{Y}_n)_{n\in\mathbb{N}},$$

the process that we call the BRW with two barriers and offspring given by a $P.P.P.(\lambda/2)$.

The survival of the process is defined as the event:

$$\mathcal{S}_\lambda := \bigcap_{n\geq 0} \{\mathcal{Y}_n \neq \emptyset\}.$$

The critical parameter and its localization

Since

$$\mathbb{P}\left(\mathcal{S}_{\lambda_1}\right) \leq \mathbb{P}\left(\mathcal{S}_{\lambda_2}\right)$$

for any $0 < \lambda_1 \leq \lambda_2$, we define the critical parameter as

$$\lambda_c\left(\mathcal{Y}\right) := \sup\{\lambda > 0 : \mathbb{P}\left(\mathcal{S}_{\lambda}\right) = 0\}.$$

Theorem (Coletti, Marić and R. 2021^{+})

$1.286814 \leq \lambda_c\left(\mathcal{Y}\right) \leq 1.287191.$

The proof is by coupling with two suitably defined multi-type branching processes!

Multi-type branching processes related to $\mathcal{Y}$

Consider, for each $m \geq 1$, the partition of $[-1, 1]$ given by:

$$\mathcal{P}_m := \{x_0^m, \ldots, x_{2^{m+1}}^m\}, \text{ with } x_j^m := -1 + j/2^m \text{ for } j \in \{0, \ldots, 2^{m+1}\}.$$

For $j \in \{1, \ldots, 2^{m+1} - 1\}$ let

$$I_j^m := [x_{j-1}^m, x_j^m), \quad \text{set } I_{2^{m+1}}^m := \left[x_{2^{m+1}-1}^m, x_{2^{m+1}}^m\right],$$

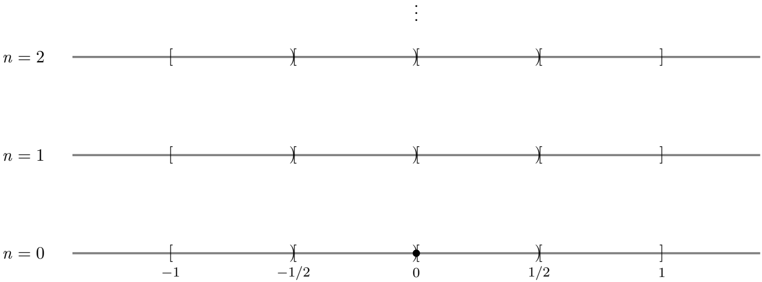
and say that a particle of $\mathcal{Y}$ located at x:

... is of type j if, and only if, $x \in I_j^m$.

The resulting process, with labeled particles, will be denoted by

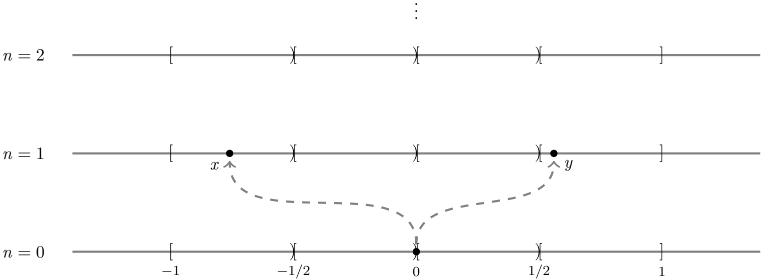
$$\mathcal{Y}^m = (\mathcal{Y}_n^m)_{n \geq 0}.$$

A process $\mathcal{X}^m$ dominated by $\mathcal{Y}^m$



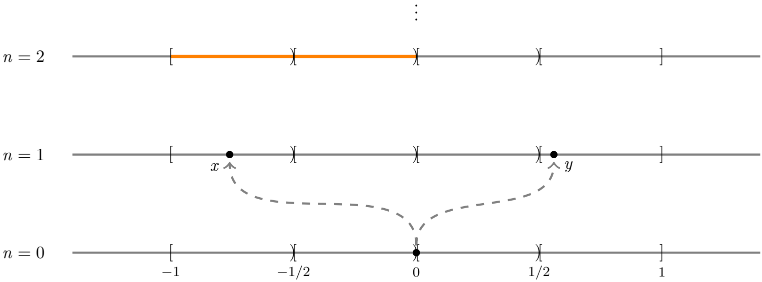
Let $m = 2$ so $I_1^2 = [-1, -1/2)$, $I_2^2 = [-1/2, 0)$, $I_3^2 = [0, 1/2)$, $I_4^2 = [1/2, 1]$.

A process $\mathcal{X}^m$ dominated by $\mathcal{Y}^m$



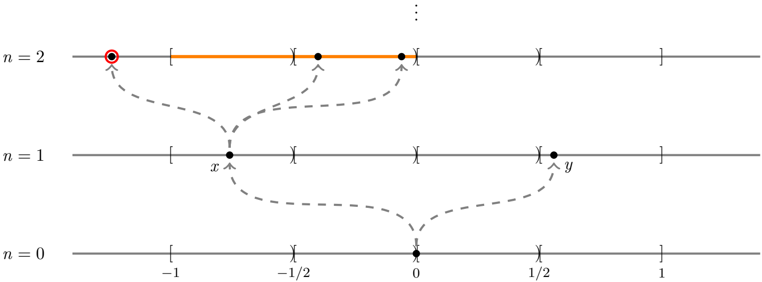
Initially let $\mathcal{X}_1^m := \mathcal{Y}_1^m$.

A process $\mathcal{X}^m$ dominated by $\mathcal{Y}^m$



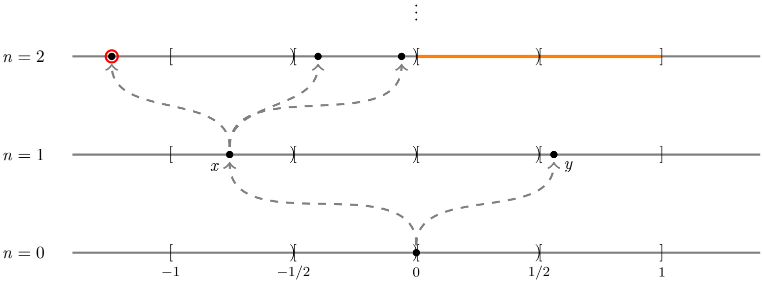
Since $x < 0$ then only offspring in $[-1, x^m_{j-1} + 1]$ belong to $\mathcal{X}^m_2$.

A process $\mathcal{X}^m$ dominated by $\mathcal{Y}^m$



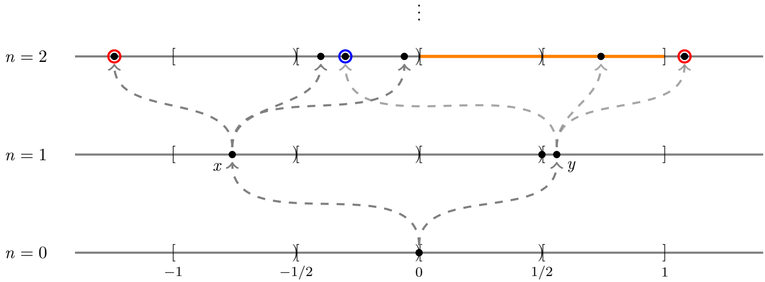
Since $x < 0$ then only offspring in $[-1, x_{j-1}^m + 1]$ belong to $\mathcal{X}_2^m$.

A process $\mathcal{X}^m$ dominated by $\mathcal{Y}^m$



But, since $y \geq 0$ then only offspring in $[x_j^m - 1, 1]$ belong to $\mathcal{X}_2^m$.

A process $\mathcal{X}^m$ dominated by $\mathcal{Y}^m$



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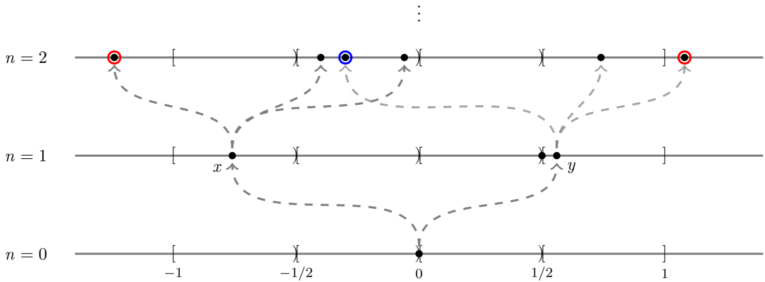
A process $\mathcal{X}^m$ dominated by $\mathcal{Y}^m$



So particles inside a blue circle belong to $\mathcal{Y}^m$ but do not belong to $\mathcal{X}^m$.

Lemma 1

Fix $m \geq 1$. Then, for any $n \geq 1$, $\mathcal{X}_n^m \subseteq \mathcal{Y}_n^m$ a.s.

The multi-type branching process with 2^m types related to $\mathcal{X}^m$, is such that the matrix of expected numbers of progeny of all types of parent participles of all types is given by

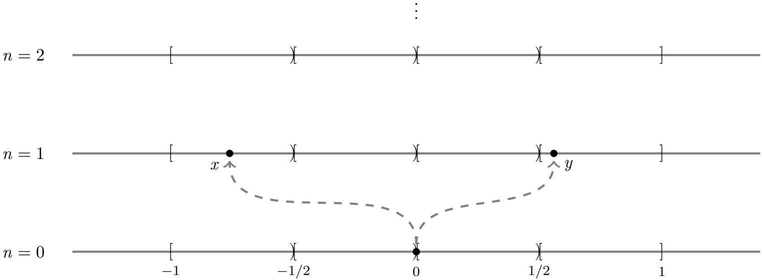
$$M(\mathcal{X}^m)(i,j) = \frac{\lambda}{2} \times \frac{1}{2^m} \times \mathbf{1}_{(|i-j| \leq 2^{m-1}-1)}.$$

If we denote by $\lambda_c\left(\mathcal{X}^m\right)$ the critical parameter of such a process, then:

Proposition 3

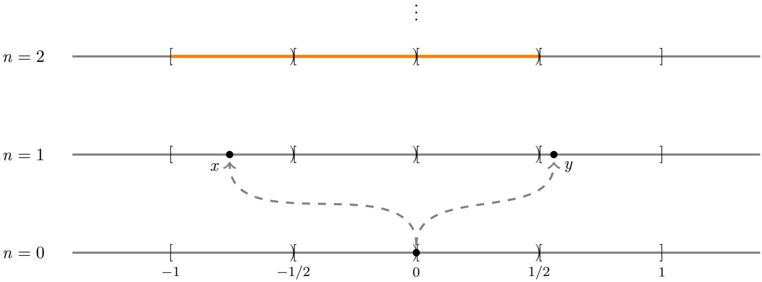
Fix $m \geq 1$. Then $\lambda_c\left(\mathcal{X}^m\right) \geq \lambda_c\left(\mathcal{Y}\right)$.

A process $\mathcal{Z}^m$ dominating $\mathcal{Y}^m$



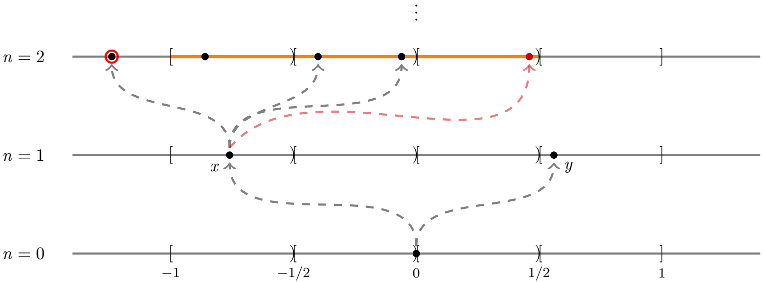
Initially $\mathcal{Z}_1^m := \mathcal{Y}_1^m$.

A process $\mathcal{Z}^m$ dominating $\mathcal{Y}^m$



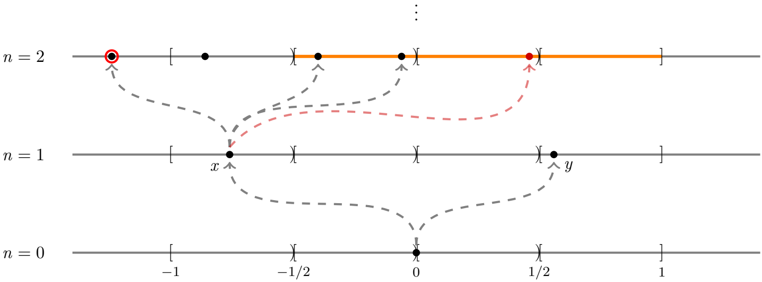
Since $x < 0$ then we allow offspring in $[-1, x_j^m + 1]$, which belong to $\mathcal{Z}_2^m$.

A process $\mathcal{Z}^m$ dominating $\mathcal{Y}^m$



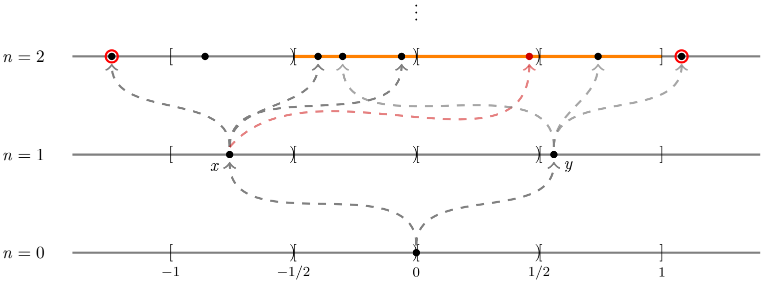
Since $x < 0$ then we allow offspring in $[-1, x_j^m + 1]$, which belong to $\mathcal{Z}_2^m$.

A process $\mathcal{Z}^m$ dominating $\mathcal{Y}^m$



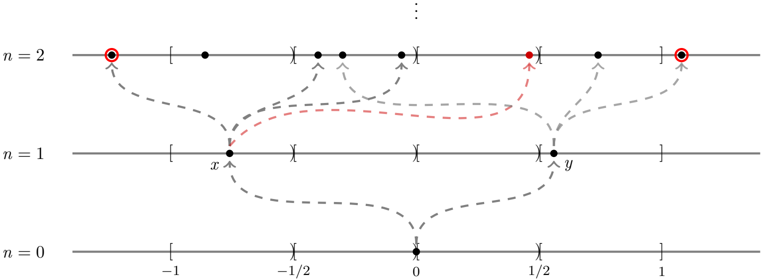
But, since $y \geq 0$ then we allow offspring in $[x_{j-1}^m - 1, 1]$, which belong to $\mathcal{Z}_2^m$.

A process $\mathcal{Z}^m$ dominating $\mathcal{Y}^m$



But, since $y \geq 0$ then we allow offspring in $[x_{j-1}^m - 1, 1]$, which belong to $\mathcal{Z}_2^m$.

A process $\mathcal{Z}^m$ dominating $\mathcal{Y}^m$



So red particles belong to $\mathcal{Z}^m$ but do not belong to $\mathcal{Y}^m$.

Lemma 2

Fix $m \geq 1$. Then, for any $n \geq 1$, $\mathcal{Y}_n^m \subseteq \mathcal{Z}_n^m$ a.s.

The multi-type branching process with 2^m types related to $\mathcal{Z}^m$, is such that the matrix of expected numbers of progeny of all types of parent participles of all types is given by

$$M(\mathcal{X}^m)(i,j) = \frac{\lambda}{2} \times \frac{1}{2^m} \times \mathbf{1}_{(|i-j| \leq 2^{m-1})}.$$

If we denote by $\lambda_c(\mathcal{Z}^m)$ the critical parameter of such a process, then:

Proposition 4

Fix $m \geq 1$. Then $\lambda_c(\mathcal{Z}^m) \leq \lambda_c(\mathcal{Y})$.

The bounds for $\lambda_c\left(\mathcal{Y}\right)$

By construction, we have:

$$\lambda_c\left(\mathcal{X}^m\right) \geq \lambda_c\left(\mathcal{X}^{m+1}\right), \text{ for } m \geq 1,$$

and

$$\lambda_c\left(\mathcal{Z}^m\right) \leq \lambda_c\left(\mathcal{Z}^{m+1}\right), \text{for } m \geq 1.$$

Thus

$$\lim_{m\to+\infty} \lambda_c(\mathcal{Z}^m) \leq \lambda_c(\mathcal{Y}) \leq \lim_{m\to+\infty} \lambda_c(\mathcal{X}^m).$$

Banded symmetric Toeplitz matrices and the λ_c's

$\lambda_c(\mathcal{X}^m)$ and $\lambda_c(\mathcal{Z}^m)$ are the maximum eigenvalue of the matrices of the mean values for the respective multi-type branching processes.
For $m = 2$, we get

$$M(\mathcal{X}^2) = \frac{\lambda}{2} \times \frac{1}{4} \times \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad M(\mathcal{Z}^2) = \frac{\lambda}{2} \times \frac{1}{4} \times \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Banded symmetric Toeplitz matrices and the λ_c's

... and for $m > 2$ their dimensions are $2^m \times 2^m$:

$$M(\mathcal{X}^m) = \frac{\lambda}{2} \times \frac{1}{2^m} \times T_{2^m, 2^{m-1}+1} \quad \text{and} \quad M(\mathcal{Z}^m) = \frac{\lambda}{2} \times \frac{1}{2^m} \times T_{2^m, 2^{m-1}+2}$$

where $T_{k,d}$ denotes the $k \times k$ banded symmetric Toeplitz matrix with $0-1$ values and bandwidth d. For example:

$$T_{7,2} = \begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 & 0 & 0 \\ t_1 & t_0 & t_1 & t_2 & 0 & 0 & 0 \\ t_2 & t_1 & t_0 & t_1 & t_2 & 0 & 0 \\ 0 & t_2 & t_1 & t_0 & t_1 & t_2 & 0 \\ 0 & 0 & t_2 & t_1 & t_0 & t_1 & t_2 \\ 0 & 0 & 0 & t_2 & t_1 & t_0 & t_1 \\ 0 & 0 & 0 & 0 & t_2 & t_1 & t_0 \end{bmatrix}$$

with $t_i = 1$ for any $i \in \{0, 1, 2\}$.

Banded symmetric Toeplitz matrices and the λ_c's

We let:

$$\lambda_c(\cdot) = \frac{2^{m+1}}{\text{Perron-Frobenius eigenvalue of } T^*(\cdot)}.$$

m	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$\lambda_c(\mathcal{X}^m)$	1.527864	1.393724	1.337647	1.311711	1.299210	1.293070
$\lambda_c(\mathcal{Z}^m)$	1.123106	1.198501	1.240855	1.263415	1.275074	1.281004

m	$m=7$	$m=8$	$m=9$	$m=10$	$m=11$	$m=12$
$\lambda_c(\mathcal{X}^m)$	1.290027	1.288513	1.287757	1.287379	1.287191	?
$\lambda_c(\mathcal{Z}^m)$	1.283995	1.285496	1.286249	1.286625	1.286814	?

Our additional questions

We want to prove:

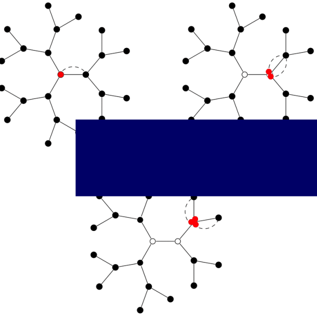
$$\lim_{m\to+\infty} \lambda_c\left(\mathcal{Z}^m\right) = \lim_{m\to+\infty} \lambda_c\left(\mathcal{X}^m\right) = \lambda_c\left(\mathcal{Y}\right).$$

... and to extend the analysis for $L > 1$!

Our reference for this model

A stochastic model and survival analysis for plant species in a reduced habitat, joint work with Coletti (UFABC) and Marić (Union University),

Preprint in preparation.



Thanks!

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