On the Convergence of the Drainage Network Model with Branching

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Drainage Network with Branching

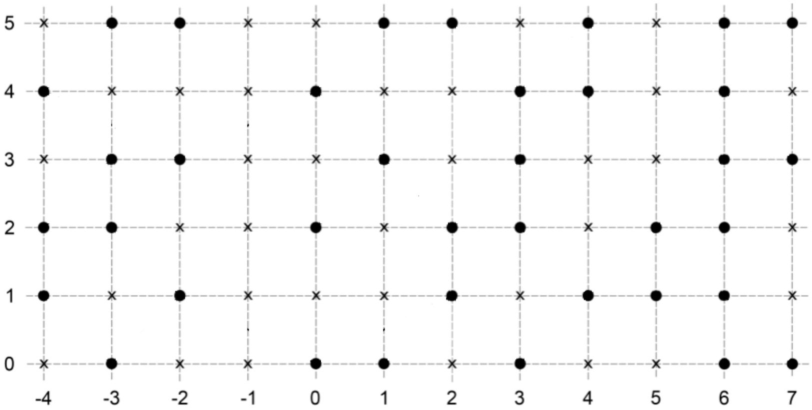
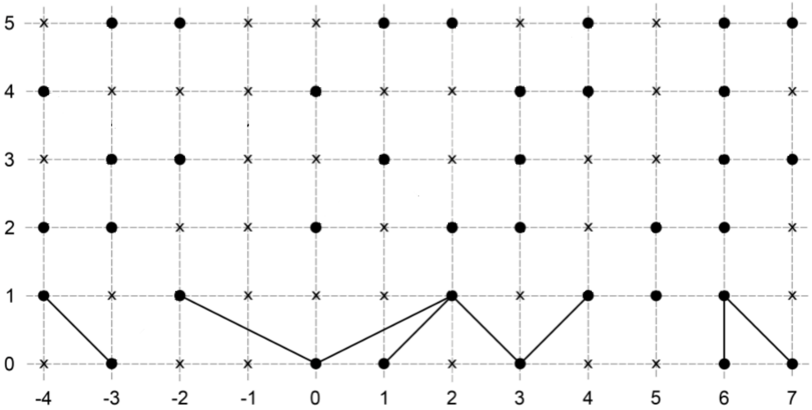


Figure: Construction of the Drainage Network with Branching

Drainage Network with Branching



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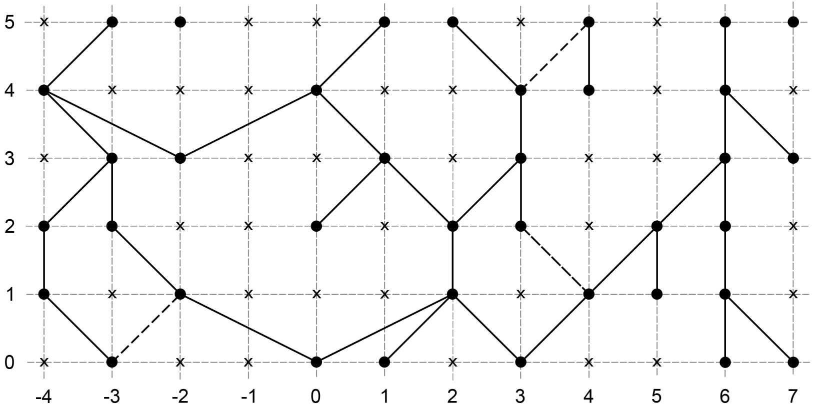

Figure: Construction of the Drainage Network with Branching

Drainage Network with Branching

The Drainage Network with Branching (DNB) will depend of two family of variables:

- $(\omega(z))_{z\in\mathbb{Z}^2}$: A family of independent Bernoulli random variables with parameter $p \in (0,1)$.

- $(\theta(z))_{z\in\mathbb{Z}^2}$: A family of independent and identically distributed random variables on $\{-1,0,1\}$ that depends on a parameter $\epsilon \in (0,1)$ and have the following probability function:

$$P(\theta(z)=0)=\epsilon, \qquad P(\theta(z)=-1)=P(\theta(z)=1)=\frac{1-\epsilon}{2}.$$

$\mathcal{X}_\epsilon$ will denote the DNB with branching parameter ϵ.

$\mathcal{X}^n_{\epsilon_n} = \{(\frac{z_1}{n}, \frac{z_2}{n^2}) \in \mathbb{R}^2 : (z_1, z_2) \in \mathcal{X}_\epsilon\}$ will denote the diffusively rescaled DNB.

$W^l_n \subset \mathcal{X}^n$ will be the subset of the l-paths.

$W^r_n \subset \mathcal{X}^n$ will be the subset of the r-paths.

Drainage Network with Branching

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If we have $\alpha \in [0, 1)$, then $\mathcal{X}^n_{\epsilon_n}$ does not converge in distribution under diffusive scaling. In this case, $n\,\epsilon_n$ diverges to infinity when $n \to \infty$, which means that the lattice $\mathbb{Z}^2$ is compressed faster than the reduction of the branching probability.

Dual process for Drainage Network with Branching.

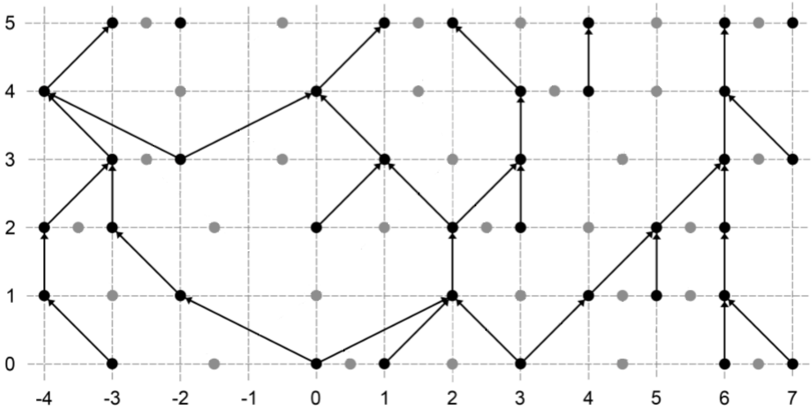

Figure: Construction of the dual process for the Drainage Network with Branching

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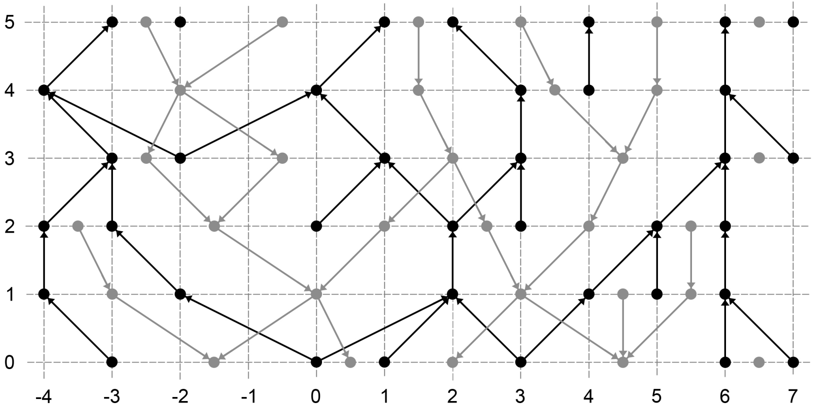

Figure: Construction of the dual process for the Drainage Network with Branching

Brownian Web

Intuitively, the Brownian Web is a collection of coalescing Brownian motions, starting from every point in space-time plane $\mathbb{R}^2$.

[Fontes, Isopi, Newman and Ravishankar] Brownian Web

There exists an $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variable $\mathcal{W}$, called standard Brownian Web, whose distribution is uniquely determined by the following properties:

(a) For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique path $\pi_z \in \mathcal{W}(z)$;

(b) For any finite deterministic set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(\pi_{z_1}, \ldots, \pi_{z_k})$ is distributed as coalescing (standard) Brownian motions;

(c) For any deterministic countable dense subset $D \subset \mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{\pi_z : z \in D\}$ in (Π, d).

Brownian Web

The Brownian Web $\mathcal{W}$ has a dual process $\widehat{\mathcal{W}}$, which is called the *dual Brownian Web*. $\widehat{\mathcal{W}}$ is a collection of coalescing paths running backward in time, which is uniquely determined by the restriction that these paths cannot cross any path from $\mathcal{W}$.

The previous characterization can be extended to allow the Brownian paths to have a fixed diffusion coefficient $\lambda^2 \neq 1$ and a drift $b \neq 0$. The only difference is in property (b), where the coalescing Brownian motions may have a diffusion coefficient distinct from one and a non-zero drift. We denote by $\mathcal{W}_{\lambda,b}$ the *Brownian Web with diffusion coefficient* $\lambda^2 > 0$ *and drift* $b \in \mathbb{R}$, We also denote $\mathcal{W}_{\lambda,b} = \mathcal{W}_\lambda$ if $b = 0$.

Brownian Net

The Brownian Net generalizes the Brownian Web by allowing paths to branch and was described by Rongfeng Sun and J. M. Swart. We will denote by $\mathcal{N}_b$ the Brownian Net with branching parameter b.

Brownian Net

Consider the paths $(l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_{k'}})$ starting from the points $(z_i)_{1\le i\le k}$ and $(z'_j)_{1\le j\le k'}$ in $\mathbb{R}^2$ that satisfy:

- $(l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_{k'}})$ evolve independently until they meet each other. The l-paths $(l_{z_1}, \ldots, l_{z_k})$ coalesce when they meet and the same is true for the r-paths $(r_{z'_1}, \ldots, r_{z'_{k'}})$.

- Each pair of left-right paths $(l_{z_i}, r_{z'_j})$ solves the following system of SDEs:

$$\begin{cases} dL_t = I_{\{L_t \neq R_t\}} dB^l_t + I_{\{L_t = R_t\}} dB^s_t - dt, \\ dR_t = I_{\{L_t \neq R_t\}} dB^r_t + I_{\{L_t = R_t\}} dB^s_t + dt, \end{cases}$$

where the l-path L and the r-path R have the restriction that $L_t \le R_t$ for all $t \ge \inf\{u \ge \sigma_L \vee \sigma_R : L_u \le R_u\}$, with σ_L and σ_R being the starting times of L and R.

The pair of SDEs above has a unique solution and the above properties uniquely determine the joint law of $(l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_{k'}})$, called *left-right coalescing Brownian motions*.

Brownian Net

[Rongfeng Sun & Swart] Left-Right Brownian Web and it's dual

There exists an $\mathcal{H}^2$-valued random variable $(\mathcal{W}^l, \mathcal{W}^r)$, called the (standard) left-right Brownian Web, whose distribution is uniquely determined by the following properties:

(i) *For each deterministic $z \in \mathbb{R}^2$, $\mathcal{W}^l$ and $\mathcal{W}^r$ almost surely contain a single path each that start from point z;*

(ii) *For any finite deterministic set of points $z_1, \ldots z_k, z'_1, \ldots z'_{k'} \in \mathbb{R}^2$, the collection of paths $(l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_{k'}})$ is distributed as a family of left-right coalescing Brownian motions.*

(iii) *For any deterministic countable dense sets $\mathcal{D}^l, \mathcal{D}^r \subset \mathbb{R}^2$,*

$$\mathcal{W}^l = \overline{\{l_z : z \in \mathcal{D}^l\}} \text{ and } \mathcal{W}^r = \overline{\{r_z : z \in \mathcal{D}^r\}} \text{ a.s.}$$

Futhermore, almost surely there exists a dual left-right Brownian Web $(\widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r) \in \widehat{\mathcal{H}}^2$, such that $(\mathcal{W}^l, \widehat{\mathcal{W}}^l)$ (resp. $(\mathcal{W}^r, \widehat{\mathcal{W}}^r)$) is distributed as $(\mathcal{W}, \widehat{\mathcal{W}})$ tilted with drift -1 (resp. $+1$).

Brownian Net

Given a set of paths K, denote by $\mathcal{H}_{cross}(K)$ the set of paths obtained by hopping a finite number of times among paths in K at crossing times. The Brownian Net is a $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variable that can be constructed by setting $\mathcal{N} = \overline{\mathcal{H}_{cross}(\mathcal{W}^l \cup \mathcal{W}^r)}$. It gives the hopping characterization of The Brownian Net.

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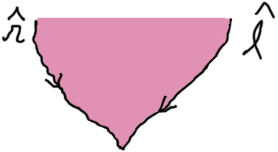
[Rongfeng Sun & Swart] Brownian Net - Wedge characterization

Let $(\mathcal{W}^l, \mathcal{W}^r, \widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r)$ be the standard left-right Brownian Web and its dual. Then almost surely,

$$\mathcal{N} = \{\pi \in \Pi : \pi \text{ does not enter any wedge of } (\widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r) \text{ from outside}\}$$

is the standard Brownian Net associated with $(\widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r)$, i.e., $\mathcal{N} = \overline{\mathcal{H}_{cross}(\mathcal{W}^l \cup \mathcal{W}^r)}$.



Drainage Network with Branching - Main results

Our main objective is to describe the asymptotic behavior of the diffusively rescaled DNB when $\epsilon_n = \mathfrak{b} n^{-\alpha}$ for different values of $\alpha > 0$.

Theorem 1:

If $\epsilon_n = \mathfrak{b} n^{-\alpha}$ for some $\mathfrak{b} > 0$ and $\alpha > 1$, then $\mathcal{X}^n_{\epsilon_n}$ converges in distribution in $\mathcal{H}$ to the Brownian Web $\mathcal{W}_\lambda$ when $n \to \infty$, where $\lambda^2 = \lambda_p^2$ is the variance of an increment of a DNB path.

Drainage Network with Branching - Main results

Theorem 2:

Let W_n^l (resp. W_n^r) be the set of l-paths (resp. r-paths) of a diffusively rescaled DNB with parameter $\epsilon_n = \mathfrak{b} n^{-1}$ for some $\mathfrak{b} > 0$. There exist $\hat{W}_n^l$ and $\hat{W}_n^r$ dual processes of W_n^l and W_n^r respectively such that

$$(W_n^l, W_n^r, \hat{W}_n^l, \hat{W}_n^r) \Longrightarrow (\mathcal{W}_{\lambda,b_p}^l, \mathcal{W}_{\lambda,b_p}^r, \widehat{\mathcal{W}}_{\lambda,b_p}^l, \widehat{\mathcal{W}}_{\lambda,b_p}^r) \;\; as \;\; n \to \infty,$$

where $b_p = \frac{\mathfrak{b}(1-p)}{(2-p)^2}$.

Drainage Network with Branching - Main results

Theorem 3:

If we have $\mathcal{X}^n_{\epsilon_n}$ the diffusively rescaled Drainage Network with branching parameter $\epsilon_n = \mathfrak{b} n^{-1}$ for $\mathfrak{b} > 0$, then $\mathcal{X}^n_{\epsilon_n}$ is tight and any subsequential limit contains the Brownian Net $\mathcal{N}_{b_p,\lambda}$ with $b_p = \frac{\mathfrak{b}(1-p)}{(2-p)^2}$.

We conjecture that $\mathcal{X}^n_{\epsilon_n}$ converges to $\mathcal{N}_{b_p,\lambda}$

Two key ingredients of the proofs

- Control on the probability of “long jumps”: For any fixed $s > 0$ and $n \in \mathbb{N}$, the probability that an specific path π from the DNB will make at least one jump of size greater than $g \geq 1$ in a time window of size sn^2 is bounded from above by $2sn^2 e^{-c(g-1)}$, for some $c = c_p > 0$.

- Estimates for coalescence times: Let τ_k denote the time until coalescence between two l-paths (or two r-paths) that start within distance k from each other. There exists a constant $C_0 > 0$ such that for every $t > 0$ and $k \in \mathbb{N}$ we have:

$$P(\tau_k > t) \leq \frac{C_0 k}{\sqrt{t}}.$$

Main steps in the proof of Theorem 1 ($\alpha > 1$)

Consider a sequence of $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variables $(\mathcal{Z}_n)_{n\in\mathbb{N}}$. If $(\mathcal{Z}_n)_{n\in\mathbb{N}}$ satisfies the following four conditions, then it converges to a Brownian Web.

(T): The law of $(\mathcal{Z}_n)_{n\in\mathbb{N}}$ is tight.

(I): There exists $\pi_{n,z} \in \mathcal{Z}_n$ for each $z \in \mathbb{R}^2$, such that for any deterministic $z_1, \ldots, z_k \in \mathbb{R}^2$, $(\pi_{n,z_i})_{1\le i\le k}$ converge in distribution to coalescing Brownian motions starting at $(z_i)_{1\le i\le k}$.

(B1*): For any $\mathcal{Z}$ a subsequential weak limit of $\mathcal{Z}_n$, we have that for all $t > 0$,

$$\limsup_{\delta\downarrow 0} \sup_{a,t_0\in\mathbb{R}} P[\eta_{\mathcal{Z}}(t_0, t; a, a+\delta) \ge 2] = 0$$

,
where $\eta_{\mathcal{X}}(t_0, t; a, b) = |\{\pi(t_0 + t) : \pi \in \mathcal{X}, \pi(t_0) \in [a, b]\}|,\ t_0 \in \mathbb{R}, t > 0, a < b$, which counts the number of distinct paths in $\mathcal{X}$ at time $t_0 + t$, among all paths that start from interval [a,b] at time t_0.

(B2*): For any $\mathcal{Z}$ a subsequential weak limit of $\mathcal{Z}_n$, we have that for all $t > 0$

$$\limsup_{\delta\downarrow 0} \frac{1}{\delta} \sup_{a,t_0\in\mathbb{R}} P[\eta_{\mathcal{Z}}(t_0, t; a, a+\delta) \ge 3] = 0.$$

Main steps in the proof of Theorem 1 ($\alpha > 1$)

(i) To prove (I), we show that under diffusive scaling, the distance between any DNB path starting from a fixed point z and the path from the usual drainage network starting from this same point and in the same environment goes to zero as $n \to \infty$.

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(ii) To prove (T), we use that condition (I) implies tightness of $(W_n^l \cup W_n^r)$ and the fact that the modulus of continuity of the entire system can be bounded by the modulus of continuity of the collection of l-paths and r-paths.

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(iii) The proofs of conditions (B1*) and (B2*) are more technical.

Main steps in the proof of Theorem 3 ($\alpha = 1$)

Consider a sequence of $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variables $(\mathcal{Y}_n)_{n\in\mathbb{N}}$. Based on the existence of subsets of non-crossing paths W_n^l, $W_n^r \subset \mathcal{Y}_n$, there are five conditions that, if satisfied, guarantees that $(\mathcal{Y}_n)_{n\in\mathbb{N}}$ converges to the Brownian Net.

(C) No path $\pi \in \mathcal{Y}_n$ crosses any $l \in W_n^l$ from right to left and no path $\pi \in \mathcal{Y}_n$ crosses any $r \in W_n^r$ from left to right.

($\mathbf{I}_{\mathcal{N}}$) There exist $l_{n,z} \in W_n^l$ and $r_{n,z} \in W_n^r$ for each $z \in \mathbb{R}^2$, such that for any deterministic $z_1, \dots, z_k, z'_1, \dots, z'_{k'} \in \mathbb{R}^2$, $(l_{n,z_1}, \dots, l_{n,z_k}, r_{n,z'_1}, \dots, r_{n,z'_{k'}})$ converges in distribution to a random vector of paths $(l_{z_1}, \dots, l_{z_k}, r_{z'_1}, \dots r_{z'_{k'}})$ that is distributed as a family of left-right coalescing Brownian motions.

Main steps in the proof of Theorem 3 ($\alpha = 1$)

(H) A.s., $\mathcal{Y}_n$ contains all paths obtained by hopping among paths in $W_n^l \cup W_n^r$ at crossing times.

($\mathbf{U}_{\mathcal{N}}^{'}$) There exist $\widehat{W}_n^l, \widehat{W}_n^r \in \widehat{\mathcal{H}}$, whose starting points become dense in $\mathbb{R}^2$ as $n \to \infty$, such that a.s. paths in W_n^l and $\widehat{W}_n^l$ (resp. paths in W_n^r and $\widehat{W}_n^r$) do not cross.

($\mathbf{U}_{\mathcal{N}}^{''}$) For any weak limit point $(\mathcal{Y}, W^l, W^r, \widehat{W}^l, \widehat{W}^r)$ of $(\mathcal{Y}_n, W_n^l, W_n^r, \widehat{W}_n^l, \widehat{W}_n^r)$ and for any deterministic countable dense set $D \subset \mathbb{R}^2$, a.s. paths in $\mathcal{Y}$ do not enter any wedge of $(\widehat{W}^l(D), \widehat{W}^r(D))$ from outside.

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(i) Conditions (C) and (H) holds by the construction of the DNB. Condition ($U'_{\mathcal{N}}$) follows from the construction of the dual of the DNB.

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(iv) Extend the proof considering more l-paths and r-paths evolving together using a construction argument.

Final discussion: On the condition $(\mathrm{U}''_{\mathcal{N}})$

We conjecture that the DNB with branching parameter $\epsilon_n = \mathfrak{b} n^{-1}$ converges to the Brownian Net under diffusive scaling. But to prove that, still remains to verify that the Brownian Net is an upper bound to any subsequential limit that we obtain in Theorem 3, which can be achieved by proving that condition $(\mathrm{U}''_{\mathcal{N}})$ holds.

So, we need to verify that for any subsequential weak limit $(X, W^l, W^r, \widehat{W}^l, \widehat{W}^r)$ of $(X_n, W_n^l, W_n^r, \widehat{W}_n^l, \widehat{W}_n^r)$ and for any deterministic countable dense set $D \subset \mathbb{R}^2$, a.s. paths in X do not enter any wedge of $(\widehat{W}^l(D), \widehat{W}^r(D))$ from outside.

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