Models selection procedures for random objects driven by context tree models

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Motivation

- A stochastic chain produce a sequence of stimuli

$X_1, X_2, \ldots, X_n$

Stochastic stimuli

$Y_1, Y_2, \ldots, Y_n$

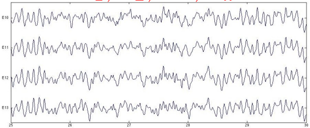
Physiological or behavioral answear

Examples

- A stochastic chain produce a sequence of stimuli

$X_1, X_2, \ldots, X_n$

$Y_1, Y_2, \ldots, Y_n$



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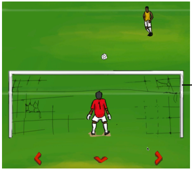

$Y_1, Y_2, \ldots, Y_n$

Game performance

https://game.numec.prp.usp.br/

Example of stochastic stimuli

Stimulus unit

- 2 - strong beat
- 1 - weak beat
- 0 - silent unit

- 2 - right
- 1 - center
- 0 - left

Example of stochastic stimuli

Stimulus unit

- 2 - strong beat
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Stimuli sequence

1. Start with a deterministic sequence

$$\mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\cdots$$

2. Replace each weak beat by a silent unit independently and with a probability $\epsilon > 0$

One example of stochastic sequence would be

$$\begin{array}{l}\mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 1\ 1\ \mathbf{2}\cdots \\ \mathbf{2}\ 1\ 0\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 0\ 0\ \mathbf{2}\ 1\ 1\ \mathbf{2}\ 0\ 1\ \mathbf{2}\cdots\end{array}$$

Characterizing the stochastic sequence

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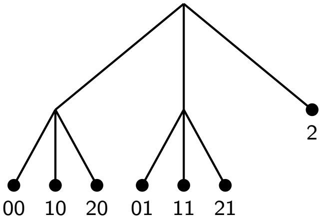

Ternary CTM

context w	p(0|w)	p(1|w)	p(2|w)
2	ϵ	$1-\epsilon$	0
21	ϵ	$1-\epsilon$	0
20	ϵ	$1-\epsilon$	0
11	0	0	1
10	0	0	1
01	0	0	1
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$\tau = \{2, 21, 11, 01, 20, 10, 00\}$:

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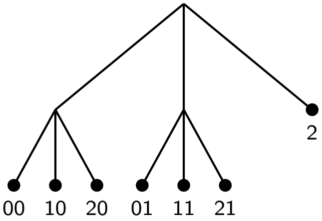

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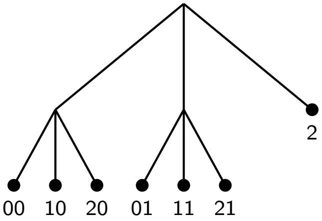

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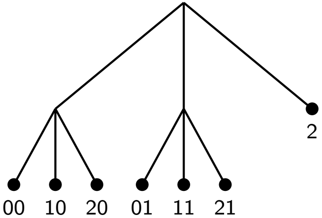

(family of transition probabilities)

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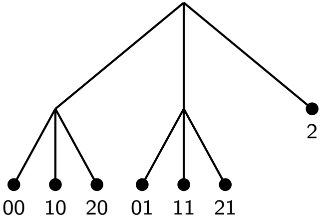

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$\tau = \{2, 21, 11, 01, 20, 10, 00\}$: (partition of the possible pasts)

- This two objects characterize the stochastic simulation algorithm

Formal definition

- The string u is said to be a suffix of v ($u \preceq v$), if there exists a string s satisfying $v = su$.

- A context tree is defined as any set $\tau \subset A^* = \cup_{m=1}^{\infty} A^m$ satisfying

(i) *Suffix Property*. No string $w \in \tau$ is a proper suffix of another string $s \in \tau$.

(ii) *Irreducibility*. No string belonging to τ can be replaced by a proper suffix without violating the suffix property.

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- We say $(X_n)_n$ is a context tree model with parameters (τ, p) if there exist a context function $c_\tau : A^* \to \tau$ such that
  (i) for any $n \geq l(\tau)$ and any finite sequence $x_{-n}^{-1} \in A^n$ such that $p(x_{-n}^{-1}) > 0$, it holds that

$$P\big(X_{n+1} = a | X_1^n = x_{-n}^{-1}\big) = p\big(X_{n+1} = a | c_\tau(x_{-n}^{-1})\big) \text{ for all } a \in A.$$

  (ii) no proper suffix of $c_\tau(x_{-n}^{-1})$ satisfies condition 2.

Context tree models

- Elements of a context tree (the leaves) are called contexts
- Introduced by Rissanen as an universal system of data compression
- Also called
  - stochastic chains with memory of variable length and
  - variable length Markov chains

Stimuli and answers relation

- **How we define the dependence of** Y_k in terms of

$$\ldots,\ X_{k-3},\ X_{k-2},\ X_{k-1},\ X_k?$$

- The distribution of Y_k should depend only on the **context** associated to this sequence of past stimulus

Formally

- The bivariate stochastic chain $(X_n, Y_n)_n$ taking values in $A \times F$ is a *sequence of random objects driven by context tree models* with parameters (τ, p, q), where $q = (q(\cdot \mid w) : w \in \tau)$ is a family of probability measures on $(F, \mathcal{F})$, if

  (i) $(X_n)_n$ is a context tree model with parameters (τ, p);

  (ii) Conditionally to the sequence $(X_n)_n$, $(Y_n)_n$ are independent random variables and, for any integers $\ell(\tau) \leq m \leq n$, any string $x_{m-\ell(\tau)+1}^n \in A^{n-m+\ell(\tau)}$ and any sequence $J_m, \ldots, J_n$ of $\mathcal{F}$-measurable sets, it holds

$$P\big(Y_m \in J_m, \ldots, Y_n \in J_n | X_{m-\ell(\tau)+1}^n = x_{m-\ell(\tau)+1}^n\big) = \prod_{k=m}^{n} q\left(Y_k \in J_k | c_\tau\left(x_1^k\right)\right)$$

What about model selection?

- For CTM we have statistical procedures to $(\hat{\tau}, \hat{p})$ consistently.
  - Algorithm context (Rissanen 83)
  - BIC (Csiszár and Talata 06)
  - SMC (Galves, Galves, García, Garcia and Leonardi 12)

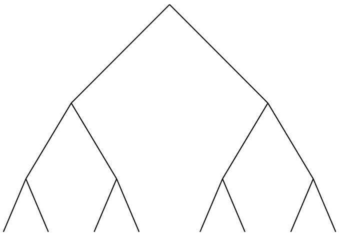
- How to check the time-dependency of the answer sequence?

- It depends on the type of answer data: functional or categorical

Model selection procedures

$$\begin{cases} \text{Functional} \Rightarrow \text{Algorithm context} + \text{Projective mehod} \\ \text{Categorical} \Rightarrow \left.\begin{cases} \text{Algorithm context} + \text{Conditional likelihood} \\ \text{Algorithm context} + \text{Offspring emp. distr.} \\ \text{BIC} \end{cases}\right\} \Rightarrow \mathit{SMC} \end{cases}$$

General algorithm context



Algorithm context

1. Set as candidate tree *the biggest* context tree possible

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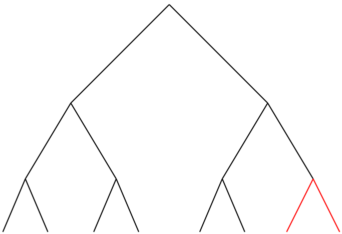

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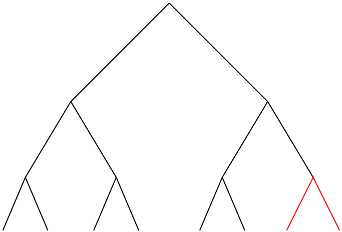

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   2.2 Use the criterion to prune or not the branch
3. Run step 2 until all branches are tested.

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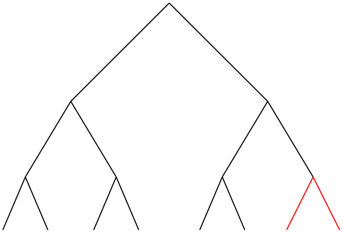

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- A subtree in τ induced by s is defined as the set $\tau_s = \{w \in \tau : s \prec w\}$.
- A subtree is said terminal if for all $w \in \tau_s$ it holds that $w = as$ for some $a \in A$.

The biggest context tree

Given a finite string $u \in A^*$, denote the number of its occurrence by

$$N_n^X(u) = \sum_{t=l(u)}^{n-1} 1_{\{X_{t-l(u)+1}^t = u\}}.$$

An admissible context tree of maximum height L for the sample $(X_1, ..., X_n)$ is any context tree τ satisfying

i) $w \in \tau$ if and only if $l(w) \leq L$ and $N_n^X(w) \geq 1$.
(any context appears at least once in the sample X)

ii) Any string $v \in A^*$ with $N_n^X(v) \geq 1$ is a suffix of some $w \in \tau$ or has a suffix $w \in \tau$. (any string which appears in the sample X is represented in τ)

The set of all admissible context trees of maximal height L: $\Gamma^L(X_1^n)$.

Functional case

- Consider $F = L^2([0, T])$ and $\mathcal{F}$ the Borel $\sigma-$algebra on $L^2([0, T])$.
- For any string $s \in A^*$ with $l(s) \leq L$, define

$$I_n(s) = \{l(s) \leq m \leq n : X_{m-l(s)+1}^m = s\}. \tag{1}$$

- By definition, the set $I_n(s)$ has $N_n^X(s)$ elements.
- Set $Y_k^{(s)}$ the $k-$occurrence of s.
- Thus $Y^{(s)} = \{Y_1^{(s)}, ..., Y_{N_n^X(s)}^{(s)}\}$ is the subsample of $(Y_1, ..., Y_n)$ induced by the string s.

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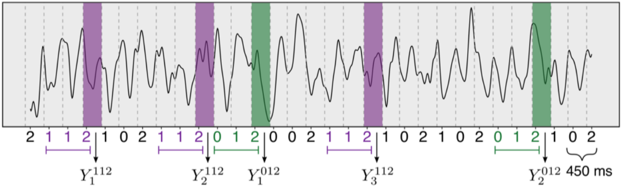

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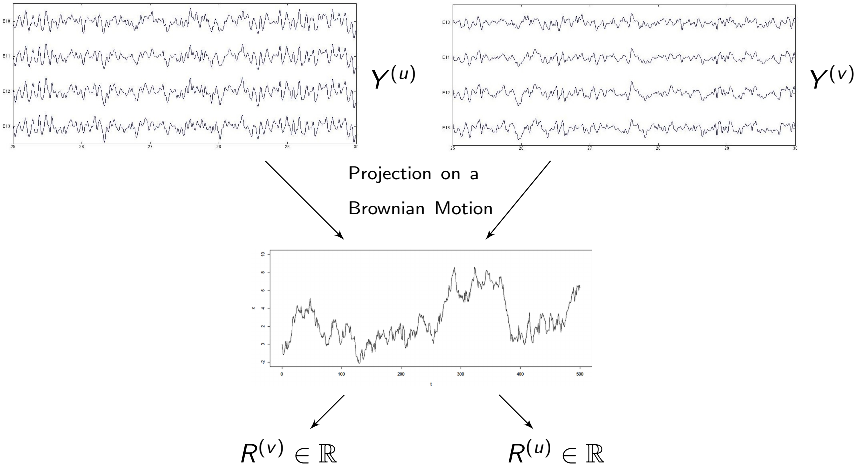
2. Choose a terminal subtree $\tau_u = \{w \in \tau : w \prec u\}$ which has not been tested yet
3. Test null hypothesis.

$$H_0^{(u)} : \mathcal{L}\left(Y_1^{(s)}, \ldots, Y_{N_n^X(s)}^{(s)}\right) = \mathcal{L}\left(Y_1^{(v)}, \ldots, Y_{N_n^X(v)}^{(v)}\right), \forall\, s, v \in \hat{\tau}_u,$$

3.1
- If the null hypothesis is not rejected, **prune** the branch.
- If the null hypothesis is rejected, **keep** the branch.

4. Run step 2 until all branches are tested.

Projective Method



Cuesta-Albertos (2006): If the law of $R_k^{(u)}$ and $R_k^{(v)}$ are different, so are the laws of $Y_k^{(u)}$ and $Y_k^{(v)}$.

Step 3.

Given a string $u \in A^*$, we test

$$H_0^{(u)} : \mathcal{L}\big(Y^{(s)}\big) = \mathcal{L}\big(Y^{(v)}\big),\ \forall\, s, v \in \hat{\tau}_u,$$

using the test statistic

$$\Delta_n(u) = \Delta_n^W(u) = \max_{s,v \in \hat{\tau}_u} \sqrt{\frac{N_n^X(s) N_n^X(v)}{N_n^X(s) + N_n^X(v)}} KS(\hat{Q}_n^{s,W}, \hat{Q}_n^{v,W}).$$

Here W is a realization of a Brownian bridge in the interval $[0, T]$, $KS\big(\hat{Q}_n^{s,W}, \hat{Q}_n^{v,W}\big)$ is the KS distance between the empirical distributions $\hat{Q}_n^{s,W}$ and $\hat{Q}_n^{v,W}$ of the projected samples

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- If $\Delta_n(u) < c$, prune the brunch
- Otherwise, keep the brunch

Hypothesis

Let P be a probability measure on $(F, \mathcal{F})$. Suppose that P satisfies
H1. The *Carleman condition*: all the absolute moments
$m_k = \int ||h||^k P(dh)$, $k \geq 1$, are finite and

$$\sum_{k \geq 1} m_k^{-1/k} = +\infty.$$

H2. *Continuity*: for any $h \in F$, where P^h is defined by

$$P^h((-\infty, t]) = P(x \in F : \langle x, h \rangle \leq t), \ t \in \mathbb{R}.$$

H3. The family q is *identifiable:* for any context $s \in \tau$ there exits a pair
$u, v \in \tau_w$ such that and $q(\cdot \mid v) \neq q(\cdot \mid u)$.

Theorem: DFGOV 19

Let $(X_0, Y_0), \ldots, (X_n, Y_n)$ be a sample produced by sequence of random objects driven by a context tree model compatible with (τ^*, p^*, q^*), and let $\hat{\tau}_n$ be the context tree selected from the sample by Algorithm with $L \geq \ell(\tau^*)$ and threshold $c_{\alpha_n} = \sqrt{(1/2)\ln(2/\alpha_n)}$, where $\alpha_n \in (0, 1)$. If (τ^*, p^*) is irreducible and q^* is continuous and satisfies Carleman condition, then for $\alpha_n \to 0$ *slowly enough* as $n \to \infty$,

$$\lim_{n\to\infty} P(\hat{\tau}_n \neq \tau^*) = 0.$$

Categorical case

- Straight adaptation from single to bivariate case.

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- Number of occurrences of the string u in the sample $(X_1, ..., X_n)$ followed by the occurrence of the symbol a in the sample $(Y_1, ..., Y_n)$

$$N_n^{XY}(u, a) = \sum_{t=l(u)}^{n-1} 1_{\{X_{t-l(u)+1}^{t}=u; Y_{t+1}=a\}}.$$

- Empirical conditional transition probability

$$\hat{q}(a|u) = \frac{N_n^{XY}(u, a)}{N_n^{X}(u)}$$

Statistic for the algorithm context

- The conditional likelihood for a sample $(X_1, Y_1), \ldots.(X_n, Y_n)$

$$L_{(\tau,\hat{q})}(Y_1^n \mid X_1^n) = \prod_{u\in\tau}\prod_{a\in A}\hat{q}(a|u)^{N_n^{XY}(u,a)},$$

- Define the statistics

$$\bar{\Delta}_n(u) = \sum_{b\in A}\sum_{a\in A} N_n^{XY}(bu,a)\log\frac{\hat{q}(a|bu)}{\hat{q}(a|u)},$$

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$$\tilde{\Delta}_n(u) = \max_{b \in A} \left(\max_{a \in A} |\hat{q}(a|u) - \hat{q}(a|bu)| \right).$$

- Fix a threshold $\delta > 0$ and denote by Δ either $\bar{\Delta}$ or $\tilde{\Delta}$

(i) If $\Delta_n(u) < \delta$ we prune the subtree $\hat{\tau}_u$.

(ii) If $\Delta_n(u) \geq \delta$ we keep $\hat{\tau}_u$.

Formally

- Set $C_w((X,Y)_1^n) = 0$ for all $w \in \mathcal{T}_n^L$, and, for any $u \prec w \in \mathcal{T}_n^L$ define

$$C_{u,n} = \max\Big\{1_{\{\Delta_n(u)\geq\delta\}}, \max_{b\in A} C_{bu,n})\Big\}.$$

The context tree estimator $\hat{\tau}_{C,n}^{\delta} = \hat{\tau}_{C}^{\delta}((X,Y)_1^n)$ is given by

$$\hat{\tau}_{C,n}^{\delta} = \{w \preceq v \in \mathcal{T}_n^L : C_{w,n} = 0 \text{ and } C_{u,n} = 1 \text{ for all } u \prec w\}.$$

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The context tree estimator $\hat{\tau}^{\delta}_{C,n} = \hat{\tau}^{\delta}_{C}((X,Y)_1^n)$ is given by

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Consistency

Let $(X_0, Y_0), \ldots, (X_n, Y_n)$ be a sample produced by sequence of random objects driven by a context tree model compatible with (τ^*, p^*, q^*). The context tree $\hat{\tau}^{\delta}_{C,n}$ selected from the sample by Algorithm, with $\ell(\tau^*) < L < \log(n)$, eventually almost surely as $n \to \infty$, satisfies

$$\hat{\tau}^{\delta}_{C,n} = \tau^*$$

Bayesian Information Criterion (BIC)

- The BIC estimator is defined as

$$\hat{\tau}^{c}_{BIC,n} = \text{argmax}_{\tau \in \Gamma^L_n} \left\{ \log L_{(\tau,\hat{q})}(Y_1^n \mid X_1^n) - c\, df(\tau) \log(n) \right\}.$$

where *df* stands for the degree of freedom of the model.

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where df stands for the degree of freedom of the model.

- For any $w \in \mathcal{T}^L_n$, define $V_{w,n} = n^{-c \cdot df(w)} L_{(w,\hat{q})}(Y_1^n \mid X_1^n)$ and the indicator $\mathcal{X}_{w,n} = 0$
- For any $w \prec u \in \mathcal{T}^L_n$ define recursively the quantity

$$V_{w,n} = \max \Big\{ n^{-c \cdot df(w)} L_{(w,\hat{q})}(Y_1^n \mid X_1^n) \,,\, \prod_{b \in A} V_{bw,n} \Big\}$$

and the indicator

$$\mathcal{X}_{w,n} = 1\Big\{ \prod_{b \in A} V_{bw,n} > n^{-c \cdot df(w)} L_{(w,\hat{q})}(Y_1^n \mid X_1^n) \Big\}.$$

Bayesian Information Criterion (BIC)

- The BIC estimator is defined as

$$\hat{\tau}^c_{BIC,n} = \text{argmax}_{\tau \in \Gamma^L_n} \Big\{ \log L_{(\tau,\hat{q})}(Y_1^n \mid X_1^n) - c\, df(\tau) \log(n) \Big\}.$$

where *df* stands for the degree of freedom of the model.

- For any $w \in \mathcal{T}^L_n$, define $V_{w,n} = n^{-c \cdot df(w)} L_{(w,\hat{q})}(Y_1^n \mid X_1^n)$ and the indicator $\mathcal{X}_{w,n} = 0$

- For any $w \prec u \in \mathcal{T}^L_n$ define recursively the quantity

$$V_{w,n} = \max \Big\{ n^{-c \cdot df(w)} L_{(w,\hat{q})}(Y_1^n \mid X_1^n) \,,\, \prod_{b \in A} V_{bw,n} \Big\}$$

and the indicator

$$\mathcal{X}_{w,n} = 1 \Big\{ \prod_{b \in A} V_{bw,n} > n^{-c \cdot df(w)} L_{(w,\hat{q})}(Y_1^n \mid X_1^n) \Big\}.$$

- The estimated context tree can be written as

$$\hat{\tau}^c_{BIC,n} = \{ w \preceq s \in \mathcal{T}^L_n : \mathcal{X}_{w,n} = 0 \text{ and } \mathcal{X}_{u,n} = 1 \text{ for all } u \prec w \}.$$

Consistency

Let $(X_0, Y_0), \ldots, (X_n, Y_n)$ be a sample produced by sequence of random objects driven by a context tree model compatible with (τ^*, p^*, q^*). The context tree $\hat{\tau}^{\delta}_{BIC,n}$ selected from the sample by BIC procedure, with $\ell(\tau^*) < L < \log(n)$, eventually almost surely as $n \to \infty$, satisfies

$$\hat{\tau}^{c}_{BIC,n} = \tau^*$$

Constant dependency

- The tree estimation procedure presented are constant dependent

$$\bar{\Delta}_n(u) = \sum_{b \in A} \sum_{a \in A} N_n^{XY}(bu, a) \log \frac{\hat{q}(a|bu)}{\hat{q}(a|u)} < \delta$$

$$\tilde{\Delta}_n(u) = \max_{b \in A} \left(\max_{a \in A} |\hat{q}(a|u) - \hat{q}(a|bu)| \right) < \delta$$

$$\text{argmax}_{\tau \in \Gamma_n^L} \left\{ \log L_{(\tau,\hat{q})}(Y_1^n |\, X_1^n) - c \cdot df(\tau) \log(n) \right\}.$$

Smallest Maximizer Criterion

(i) Construct a set of candidate models: champion trees.

(ii) Choose a optimal model within the set of champion trees.

The champion trees obtained will depend on
the model selection procedure

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Compute the champion trees:

Denote by ℓ the constant either in the algorithm context or BIC.

Start with $\ell = 0$ and successively increase the value of ℓ

Theorem: General SMC

Assume $(X_1, Y_1), \cdots, (X_n, Y_n)$ is a sample of a sequence of random objects driven by a context tree model with parameters (τ^*, p^*, q^*). Given a set $\mathcal{C}_n \subset \Gamma^L(X_n^1)$, with $|\tau^*| \leq L < \log(n)$, satisfying

(i) $\mathcal{C}_n$ is totally ordered with respect to $\succ$ and

(ii) eventually almost surely $\tau^* \in \mathcal{C}_n$ as $n \to \infty$.

The following holds:

1. For any $\tau \in \mathcal{C}_n$, with $\tau \prec \tau^*$, there exists a constant $c(\tau^*, \tau) > 0$ such that
$$\log L_{(\tau^*, \hat{q})} - \log L_{(\tau, \hat{q})} \geq c(\tau^*, \tau) n \tag{2}$$
2. For any $\tau \prec \tau' \in \mathcal{C}_n$, with $\tau^* \preceq \tau$, there exists a constant $c(\tau', \tau) > 0$ such that
$$\log L_{(\tau', \hat{q})} - \log L_{(\tau, \hat{q})} \leq c(\tau', \tau) \log n \tag{3}$$

The tree model selection procedures satisfy the Thm

Assume $(X_1, Y_1), \cdots, (X_n, Y_n)$ is a sample of a sequence of random objects driven by context tree model with parameters (τ^*, p^*, q^*). Consider the map $\ell \in [0, +\infty] \mapsto \hat{\tau}_n^\ell \in \Gamma^L(X_n^1)$ with $|\tau^*| \leq L < \log(n)$ and $\hat{\tau}_n^\ell$ denoting either $\hat{\tau}_{\bar{C}}^\ell((X, Y)_1^n)$, $\hat{\tau}_{\tilde{C}}^\ell((X, Y)_1^n)$ or $\hat{\tau}_{BIC}^\ell((X, Y)_1^n)$ Denote by

$$\mathcal{C}_n = \{\hat{\tau}_n^\ell : \ell \in [0, +\infty]\}.$$

The set $\mathcal{C}_n$ is totally ordered with respect to $\succ$ and eventually almost surely $\tau^* \in \mathcal{C}_n$ as $n \to \infty$.

Model selection procedures future work

$$\begin{cases} \text{Functional} \Rightarrow \begin{cases} \text{Algorithm context} + \text{Projective mehod} \\ \text{Algorithm context} + \text{Other statistics} \end{cases} \\ \text{Categorical} \Rightarrow \left.\begin{cases} \text{Algorithm context} + \text{Conditional likelihood} \\ \text{Algorithm context} + \text{Offspring emp. distr.} \\ \text{BIC} \\ \text{Other Statistics} \end{cases}\right\} \Rightarrow \mathit{SMC} \end{cases}$$

- **Functional case:** Duarte, A., Fraiman, R., Galves, A., Ost, G., and Vargas, C. D., 2019, Retrieving a context tree from eeg data. Mathematics, 7(5).

- **Application in neurobiological data:**
- **Categorical case:** SeqROCTM: A Matlab toolbox for the analysis of Sequence of Random Objects driven by Context Tree Models.

- **Codes:** https://github.com/noslenh/SeqROCTM-Matlab-Toolbox.