On random processes with variable length

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Fixed length

In most of the theoretical research about the random systems with inter-agency components, the set of sites, also called "of space", does not change in the process of interaction. Let's call these particles operators and processes of **fixed length**. processes where sites can not be created or deleted. Several random processes well known in the art. For example, contact processes, exclusion processes, voting model, etc.

Variable length

Here, we study a not-standard theory. Different from the traditional ones, in our process the space changes during the evolution process. Let's call these particles operators and processes of **variable length**. This class of processes can be found on the literature under other denotation, namely: *Variable or complex architecture*[1]

[1] MAES, C.. *New trends in interacting particle systems.* Markov Proc. Rel. Fields, v. 2, p. 283-288, 2005.

Variable length

Similar processes are present in Malyshev's work[2], which is motivated by some connections between computer science and quantum gravity. Also, W. Li 's work[3] has proposed some models to understand the scaling properties and the long-range correlations found in real DNA sequences.

[2] MALYSHEV, V. *Quantum Grammars.* Journal of Mathem. Physics, v. 41, n. 7, pp. 4508-4520, 2000.

[3] W. Li.*Expansion-modification systems: A model for spatial 1/f spectra*, Physical Review A 43, (10),5240–5260,1991.

Some new questions

- Establish the concept of coupling;
- Nonlinear Markov processes.

Basic statements

Let us denote by $\mathbb{A}$ the discrete topology on $\mathcal{A}$(finite set). We consider probability measure on the σ-algebra $\mathbb{A}^{\mathbb{Z}}$ on the product space $\mathcal{A}^{\mathbb{Z}}$ endowed with the topology - product of discrete topologies on all the copies of $\mathcal{A}$. We denote by $\mathcal{M}$ the set of uniform probability measures in $\mathcal{A}^{\mathbb{Z}}$.

Self-avoiding words

At this point, we will bring some definitions[4]

Given two words $W = (a_1, \dots, a_m)$ and $V = (b_1, \dots, b_n)$, where $|W| \leq |V|$, we call the integer number in the interval $[0, n-m]$ *positions* of W in V. We say that W *enters* V at a position k if

$$\forall \;\; i \in \mathbb{Z} : 1 \leq i \leq m \Rightarrow a_i = b_{i+k}.$$

[4]TOOM, A.; RAMOS, A..; ROCHA, A.; SIMAS, A. *Random Processes with Variable Length. 28º Colóquio Brasileiro de Matemática*, IMPA, 2011.
ROCHA, A.; SIMAS, A.; TOOM, A. *Substitution operators.* Journal of Statistical Physics, v. 148, n. 3, p. 585-618, 2011.

Self-avoiding words

We call a word W *self-overlapping* if there is a word V such that $|V| < 2 \cdot |W|$ and W enters V at two different positions. A word is called *self-avoiding* if it is not self-overlapping. In particular, the empty word, every word consisting of one letter and every word consisting of two different letters are self-avoiding.

Self-avoid words are not rare

Due to Svante Janson(2007) in a private communication. It is known that self-avoiding words are not very rare: in fact, for any alphabet with at least two letters the number of self-avoiding words of length n divided by the number of all words of length n tends to a positive limit when $n \to \infty$ and this limit tends to one when the number of letters in the alphabet tends to infinity.

Variable-Length: an informal introduction

A generic *substitution operator* acts from $\mathcal{M}$ to $\mathcal{M}$ as follows: given two words G and H, where G is self-avoiding, and a real number $\rho \in [0,1]$, a generic substitution operator, informally speaking, substitutes every entrance of the word G in a long word by the word H with a probability ρ or leaves it unchanged with a probability $1-\rho$ independently of states of all the other components. We denote this operator by $(G \xrightarrow{\rho} H)$.

Variable-Length: an informal introduction

We show some basic operators.

- **Conversion** $(g \xrightarrow{\rho} h)$ is the only linear operator in our list.
- **Insertion** $(\Lambda \xrightarrow{\rho} h)$. It means that a letter h is inserted with probability ρ between every two neighbor letters independently from other places.
- **Deletion** $(g \xrightarrow{\rho} \Lambda)$. It means that each occurrence of g disappears with probability ρ or remains unchanged with probability $1-\rho$ independently from the other occurrences.
- **Compression**$(G \xrightarrow{1} h)$. The word G is non-empty and self-avoiding in the alphabet.
- **Decompression** $(g \xrightarrow{1} H)$. The word H is non-empty self-avoiding in the alphabet.

Variable-Length: an informal introduction

Let $\nu \in \mathcal{M}$ and $\mathsf{P} = \mathsf{P}_1 \circ \cdots \circ \mathsf{P}_j$, where $\mathsf{P}_1, \ldots, \mathsf{P}_j$ be a finite sequence of substitution operator. Then we define the *generalized discrete substitution process* ν_t, where $\nu_0 = \nu$, as follows:

$$\nu_t(W) = \nu \mathsf{P}^t(W) \qquad \text{for every word } W.$$

and P^t denotes the t-th composition of operator P.

About invariant measure

▸ NewEstimation

Toom et. al., (2011)

Let us consider a generalized discrete substitution process $\nu_n = \nu \mathsf{P}^n$*, where* $\mathsf{P} = \mathsf{P}_1 \mathsf{P}_2 \cdots \mathsf{P}_j$ *as in above definition. Let* $R \subset \mathbb{A}^{\mathbb{Z}}$ *be some subset of* σ*-algebra* $\mathbb{A}^{\mathbb{Z}}$*. Then, if* $\nu_n(c) \leq \delta$ *(respectively* $\nu_n(c) \geq \epsilon$*) for all* $c \in R$*, then* P *has an invariant measure* μ *such that* $\mu(c) \leq \delta$ *(respectively* $\mu(c) \geq \epsilon$*) for every* $c \in R$*, where* $\delta, \epsilon > 0$ *are positive constants.*

Flip-Annihilation

The process, which we will show here is called **Flip-Annihilation**[5], it has discrete-time and its states are bi-infinite sequences, whose particles take only two values, denoted as minus and plus.

[5] TOOM, A. *Non-ergodicity in a 1-D particle process with variable length.* Journal of Statistical Physics, v. 115, n. 3-4, p. 895-924, 2004.

Flip-Annihilation

- The first one is called *flip*, under its action a component in state minus turns into a plus with probability β.

$$\mathsf{F}_\beta : \ominus \to \oplus$$

- The second one is called *annihilation*, under its action, whenever a component in state plus is a left neighbor of a component in state minus, both components disappear with probability α.

$$\mathsf{A}_\alpha : \oplus\ominus \to \Lambda$$

Flip-Annihilation

We denote

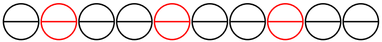
$$\mu_t = \delta_{\ominus}(\mathsf{F}_\beta \mathsf{A}_\alpha)^t, \tag{1}$$

where $\delta_{\ominus}$ is the measure concentrated in the configuration all minus, whose all particles assume the state minus.

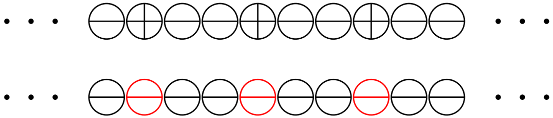
An illustration

$$\cdots \;\ominus\ominus\ominus\ominus\ominus\ominus\ominus\ominus\ominus\ominus\; \cdots \qquad t = 0$$

An illustration

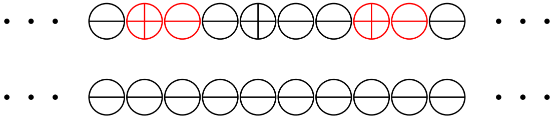
$\cdots$  $\cdots$ $\quad t = 0$

An illustration



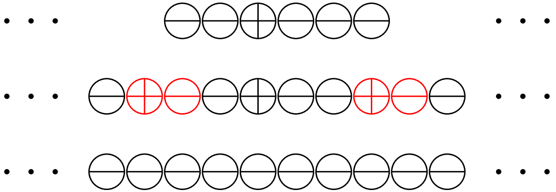
$t = 0$

An illustration



$t = 0$

An illustration



$t = 1$

$t = 0$

NewEstimation SNewEstimation

Theorem (FA1)

For all natural t the frequency of pluses in the measure μ_t does not exceed $250 \cdot \beta/\alpha^2$.

Theorem 1 was improved[6], previous we had 300 instead of 250. Since $\delta_\oplus$ is invariant for this process, theorem 1 implies that the operator cannot be ergodic whenever $\beta < \alpha^2/250$, because in this case μ_t cannot tend to $\delta_\oplus$.

[6] Ramos, A. and TOOM A. *An error correction. Letter to the editor.* Journal of Statistical Physics, **131**, n. 1, april 2008, pp. 167-168.

A phase transition

Theorem (FA2)

If $2\,\beta > \alpha$, the measures μ_t tend to $\delta_\oplus$ when $t \to \infty$.

Theorems 1 and 2 shows that the sequence of measures μ_t has at least two different modes of behavior. In one mode ($\beta > \alpha/2$) these measures tend to $\delta_\oplus$ when $t \to \infty$ and in the other mode ($\beta < \alpha^2/250$) they do not tend to $\delta_\oplus$.

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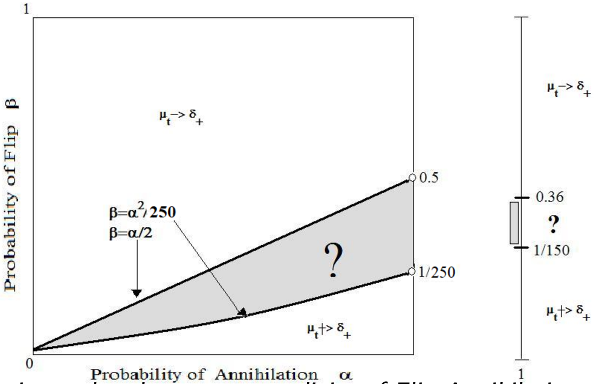


Figure: *A scheme of our main results about non-ergodicity of Flip-Annihilation process in the cases* $\alpha < 1$ *and* $\alpha = 1$.

A troublesome case ($\alpha = 1$)

It is sufficient to prove that the process μ_t is defined when $\alpha = 1$. Let us denote by μ_{chess} the (unique) measure in $\mathcal{M}$ defined by the condition

$$\mu_{chess}(\ominus,\ \oplus) = \mu_{chess}(\oplus,\ \ominus) = 1/2. \tag{2}$$

The operator Ann_1 cannot be applied to μ_{chess}. However, Ann_1 can be applied to all the other measures in $\mathcal{M}$. Thus, to include the case $\alpha = 1$, we proved that we do not apply Ann_1 at μ_{chess} in the course of inductive generation of measures μ_t.

Numerical study

Some numerical studies[7] have been performed on this process. Among the results described there, the mean-field approximation and Monte Carlo simulation (abbreviated to M.C.) gave us estimations of the transition curve, case exist, between the regions of ergodicity vs. non-ergodicity of the process.

[7] RAMOS, A. and TOOM A. *Chaos and Monte Carlo Approximations of the Flip-Annihilation process.* Journal of Statistical Physics, v. 133, n. 4, p. 761-771, 2008.

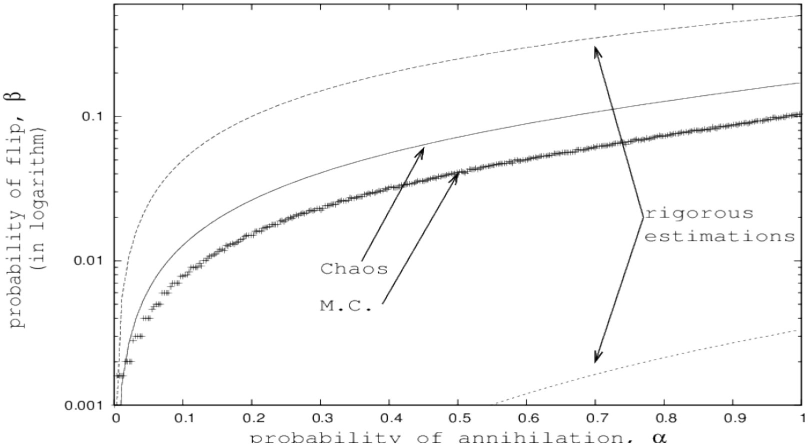


Figure: *This graph shows both rigorous estimations and the two approximations: the mean-field approximation (Chaos) and the Monte Carlo approximation (M. C.).*

In recent work[8], we consider a set of initial measures to this process. For these measures, we show the upper bound for the meantime of convergence, which is a function of the initial measure. Moreover, we obtain an upper bound to the mean quantity of minuses on the process in each time step. We improved the Theorem 1 ▸ OldEstimation that deal with the non-ergodicity of the process.

[8] Costa L. and Ramos A. *Dynamic aspects of the flip-annihilation process.* J. Math. Phys. 61, 053301, 2020.

Statements

- Lets $\mathcal{A}^{\mathbb{Z}} = \{\ominus, \oplus\}^{\mathbb{Z}}$ and $x, y \in \mathcal{A}^{\mathbb{Z}}$. We say that two configurations x and y are *close* to each other if the set $\{i \in \mathbb{Z} : x_i \neq y_i\}$ is finite.
- A configuration is called a *island of minus* if it is close to the configuration "all plus", and we denote the set of island of minus by Δ.
- Given $x \in \Delta$, we define the *population* of x, and denote by $\text{Pop}(x)$, as the quantity of minus in the island x, this is,

$$\text{Pop}(x) = \#\{i \in \mathbb{Z} : x_i = \ominus\},$$

  where $\#(\cdot)$ denotes the cardinality of the set.

Statements

We denote by $\mathcal{A}_{\ominus}$ the set of normalized measures in the countable set, Δ. A measure belonging to $\mathcal{A}_{\ominus}$ is called an *archipelago of minus*. From now on, case do not stated, μ denotes an archipelago of minus.

Statements

The operators $\mathsf{F}_\beta : \mathcal{A}_\ominus \to \mathcal{A}_\ominus$ and $\mathsf{A}_\alpha : \mathcal{A}_\ominus \to \mathcal{A}_\ominus$ are well defined. Thus, if μ is fixed, it is possible to define

$$\tau_\mu = \inf \left\{ t \geq 0 : \mu(\mathsf{F}_\beta \mathsf{A}_\alpha)^t = \delta_\oplus \right\}. \tag{3}$$

The infimum of the empty set is ∞. The quantity τ_μ denotes the time to attain the measure concentrated on configuration "all plus" having $\mathsf{F}_\beta \mathsf{A}_\alpha$, started on μ.

Statements

Lets δ_x the distribution concentrated in x and $\delta_x(\mathsf{F}_\beta\mathsf{A}_\alpha)^t$ the distribution of x^t, we define the *population of x in the time t* by $\mathsf{Pop}(x^t)$. This random variable represents the quantity of minus on the island x on the t-th time step. Given $\mu \in \mathcal{A}_\ominus$, we define the *maximum population of μ in the time t* by

$$\mathsf{M}(\mu, t) = \max\left\{\mathsf{Pop}\left((x^i)^t\right) : \mu\mathsf{P}^t = \sum_{i=1}^{\infty} k_i \delta_{(x^i)^t}\right\},$$

where $(x^i)^t$ denotes the i-th island in time t and if there is no such maximum, we say that $\mathsf{M}(\mu, t) = \infty$. We denote $\mathsf{M}(\mu, 0)$ by $\mathsf{M}(\mu)$. Note that $\mu = \sum_{i=1}^{\infty} k_i \delta_{x^i}$, where $k_1 > 0, k_2 > 0, \ldots;\ k_1 + k_2 + \cdots = 1$ and $x^i \in \Delta$ for $i \in \mathbb{N}$.

Remarks

It is clear for a δ-measure concentrated in a configuration $x \in \Delta$: if $\alpha > 0$ or $\beta > 0$ we have that $\delta_x(\mathsf{F}_\beta \mathsf{A}_\alpha)^t$ tends to $\delta_\oplus$ when t tends to infinity. However, it is not clear how fast is this convergence and how this quantity of minus behaves in the evolution of the system. Some of our Theorems provide us with information in this direction.

Numerical

Theorem (1)

Given $\mu \in \mathcal{A}_{\ominus}$ *and* τ_μ *as defined in (3). If* $\mathsf{M}(\mu)$ *is finite then*

$$\tau_\mu \leq \begin{cases} f_\mu(\alpha), & \text{if } \alpha \geq \dfrac{\mathsf{M}(\mu)}{g_\mu(\beta)}, \\[2ex] g_\mu(\beta), & \text{if } \alpha < \dfrac{\mathsf{M}(\mu)}{g_\mu(\beta)} \end{cases}$$

where $f_\mu(\alpha) = \dfrac{\mathsf{M}(\mu)}{\alpha}$ *and* $g_\mu(\beta) = \displaystyle\sum_{i=1}^{\mathsf{M}(\mu)} \frac{1}{1-(1-\beta)^i}$.

Main results

Theorem (2)

Given natural value t and $\mu \in \mathcal{A}_{\ominus}$. If $\mathsf{M}(\mu, t)$ is finite then

$$\mathbb{E}\left(\mathsf{M}(\mu, t)\right) \leq (1-\beta)^{t-1}\left(\mathsf{M}(\mu) + \alpha\beta^{\mathsf{M}(\mu)} - \alpha\right).$$

Main results

OldEstimation

Theorem (4)

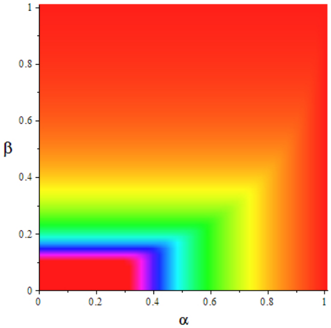
For $\alpha \in (0,1)$. *If* $\beta < \dfrac{9\alpha^2}{1000}$ *then*

(A. 5) *for all natural value* t, $\mu_t(\oplus) < 1$;

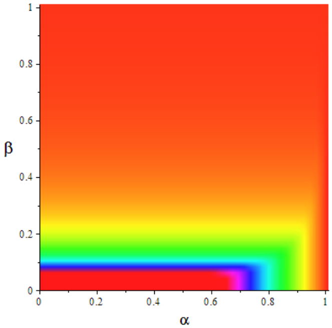
(B. 5) *there is a measure,* ν, *such that* $\nu(\mathsf{F}_\beta \mathsf{A}_\alpha) = \nu$, *where* $\nu(\oplus) < 1$. InvMeasure

Computational study

Theorem.



(a) Upper bounds of τ_μ with Pop$(x) = 10$, the minimum mean time equal to 10.0000 and the maximum mean time equal to 296.4759.



(b) Upper bounds of τ_μ with Pop$(x) = 100$, the minimum mean time equal to 100.0000 and the maximum mean time equal to 570.3338.

Computational study

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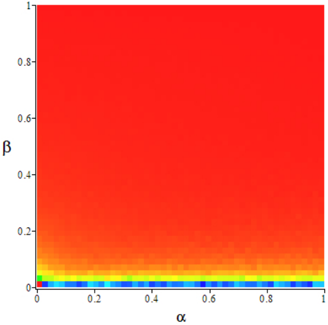
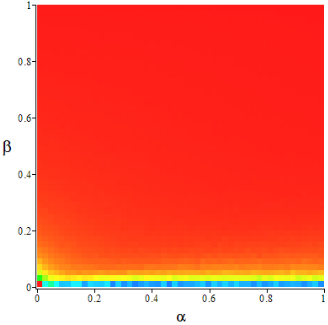


(c) The computation estimation of τ_μ with Pop$(x) = 10$, the minimum mean time equal to 1.0101 and the maximum mean time equal to 285.6263.



(d) The computation estimation of τ_μ with Pop$(x) = 100$, the minimum mean time equal to 1.0101 and the maximum mean time equal to 519.0707.

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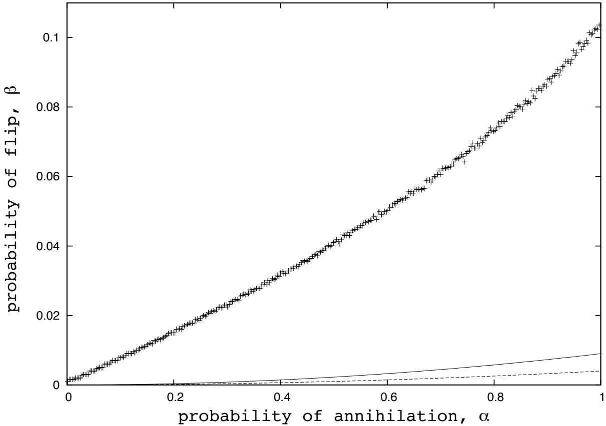


Figure: Here, we describe the curves in which $\beta = \alpha^2/250$ and $\beta = 9\alpha^2/1000$. Also, we plot the datas obtained by computational simulation of the Flip-Annihilation process. These data estimate the critical curve.

Unsolved problems

- For $\alpha, \beta \in (0,1)$ fixed. If $\mu_t(\oplus) \to 1$ when $t \to \infty$, then $\nu(\mathsf{F}_\beta \mathsf{A}_\alpha)(\oplus) \to 1$ when $t \to \infty$ for any $\nu \in \mathcal{M}$.
- Let $\alpha \in (0,1]$ be fixed and values β_1 and β_2 belongs to $(0,1)$. If $\beta_1 < \beta_2$, then

$$\delta_\ominus(\mathsf{F}_{\beta_1}\mathsf{A}_\alpha)^t(\oplus) \leq \delta_\ominus(\mathsf{F}_{\beta_2}\mathsf{A}_\alpha)^t(\oplus).$$

Flip-Murder

Motivated by the results obtained on the **Flip-Annihilation process**, we propose[9] a new process, which we denote by **Flip-Murder**. Like the first one, it has discrete-time and its states are bi-infinite sequences, whose particles take only two values, denoted here as *minus* and *plus*. Our operator is a composition of the following two operators.

[9] this work is under development with L. T. Costa and P. M. Rodriguez. However, part of these results can be found in the PHD Thesis(in Protuguese) defended on january 28, 2020 by Leon Tarquino da Costa, under the title *Dinâmica de evolução em processos com operadores de substituição.* PPGE - UFPE.

Flip-Murder

- The first one is called *flip*, under its action a component in state minus turns into a plus with probability β.

$$\mathsf{F}_\beta : \ominus \to \oplus$$

- The second one is called *murder*, under its action, whenever a component in state plus is a left neighbor of a component in state minus, this components on the state plus disappear with probability α.

$$\mathsf{M}_\alpha : \oplus\ominus \to \ominus$$

A brief reflection

Using the operator murder, we can propose a *totally asymmetric coalescence process*. The particle on the state minus attracts only the particle on its left side. In case this neighbor is on state plus, the merging of these two particles into one will happen with probability α. The new particle formed will assume the state minus.

Flip-Murder

We denote

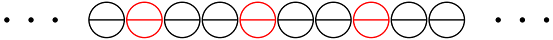
$$\mu_t = \delta_{\ominus}(\mathsf{F}_\beta \mathsf{M}_\alpha)^t, \tag{4}$$

where $\delta_{\ominus}$ is the measure concentrated in the configuration all minus, whose all particles assume the state minus.

An illustration

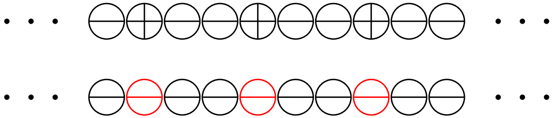
$$\cdots \quad \ominus\ominus\ominus\ominus\ominus\ominus\ominus\ominus\ominus\ominus \quad \cdots \qquad t = 0$$

An illustration



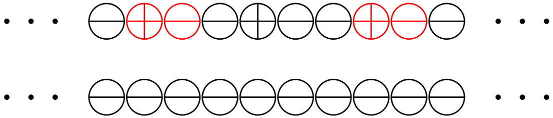
$$t = 0$$

An illustration



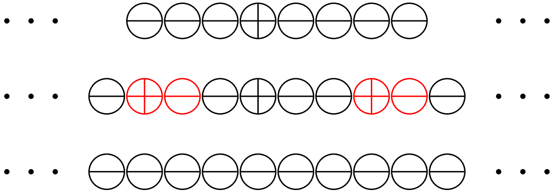
$$t = 0$$

An illustration



$t = 0$

An illustration



$t = 1$

$t = 0$

Main results

Theorem (1)

Lets $\alpha \in (0,1)$. *If* $\beta < f(\alpha)$, *then*

(A. 1) *for all nature value* t, $\mu_t(\oplus) < 1$.

(B. 1) *there is a measure,* ν, *such that* $\nu(\mathsf{F}_\beta \mathsf{M}_\alpha) = \nu$, *where* $\nu(\oplus) < 1$.

Main results

Theorem (2)

Lets $h : [0,1] \to [0,1]$. *If* $\alpha < h(\beta)$, *then the measure* μ_t *tend to* $\delta_{\oplus}$, *when* $t \to \infty$.

A phase transition

Theorems 1 and 2 shows that there is some kind of phase transition for (4).

Computational study

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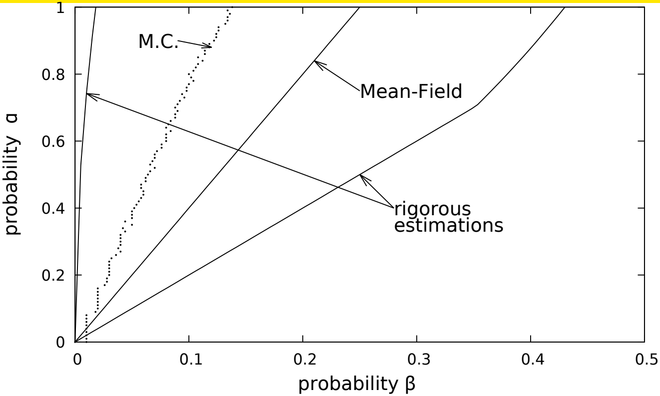

Figure: This graph shows both rigorous estimations and the two approximations for the curve which separete the ergodic and non-ergodic regions, the Mean Field Approximation (Mean-Field) and the Monte Carlo Approximation (M.C.). Every point of the M.C. curve was obtained as an average of 5 independent experiments.

Question

It is natural to ask about the kind of phase transition we get. The following theorem gives us this answer.

Answer

Theorem (3)

Let us denote by $s(\beta, \alpha)$ *the supremum of density of* $\oplus$ *in measure* μ_t *for all natural* t. *For every* $\alpha \in \left(\frac{\beta}{(1-\beta)}, 1\right)$, $s(\beta, \alpha)$ *is not continuous as a function of* β.

Computational study

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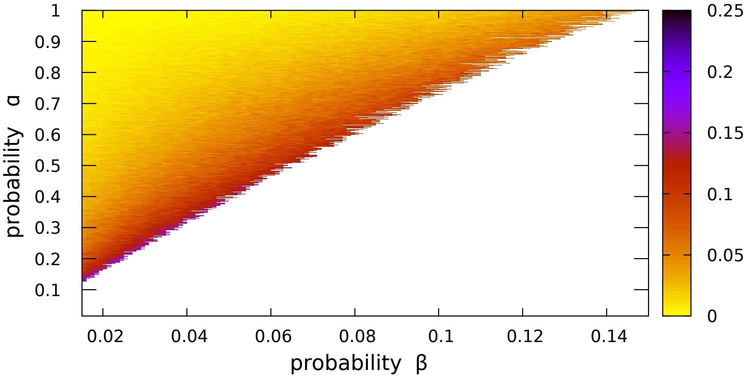


Figure: Here we used colors to represent the values of $\overline{s(\beta,\alpha)}$ in the region, where the process is suggested to be non-ergodic. The color box on the right side shows how colors from yellow to black represent the values of $\overline{s(\beta,\alpha)}$. When $\overline{s(\beta,\alpha)} = 1$, the associated color is white.

Some proofs' ideas

To prove theorem 3, we use similar ideas from that presented by (Toom, A.(2004)). There, it handles two well-known ideas: Peierls' contour method and duality planar graphs[10], which were used in the contact processes[11] and Stavskaya processes[12]. Here, the central idea is to count the probability to obtain a closed contour. ▸ Proof

[10] GRIMMETT, Geoffrey. *What is percolation?* In: Percolation. Springer, Berlin, Heidelberg, p. 1-31, 1999.
[11] HARRIS, T. E. *Contact interactions on a lattice*. Ann. Prob. 2(6), p. 969-988, 1974.
[12] TOOM, André. *A family of uniform nets of formal neurons.* Soviet Math. (Doclady), vol. 9, n. 6, pp. 1338-1341, 1968.

Variable-Length Analog of Stavskaya Process

The Variable-Length Analog of Stavskaya Process [13] was motivated by the following question: To which extent may we rely upon computer simulation in the study of random processes? Importance of this question is undeniable.

[13] Ramos, A.; Silva, F. , Sousa C. and Toom, A. *Variable-length analog of Stavskaya process: A new example of misleading simulation.* J. Math. Phys. 58, 053304, 2017.

Variable-Length Analog of Stavskaya Process

The goal of this process is to provide an example of a new kind, namely a variable-length process, to illustrate a seeming conflict between computer simulation of random processes and their theoretical study.

Like the first two processes, it has discrete-time and its states are bi-infinite sequences, whose particles take only two values, denoted here as *minus* and *plus.* Our operator is a composition of the following two operators.

Variable-Length Analog of Stavskaya Process

- The first one is called *birth*, Under its action, between any two neighbor particles there may appear a new particle in the state "plus" with probability β independently from what happens at other places.

$$\mathsf{B}_\beta : \ominus \to \oplus$$

- The second one is called *murder*, under its action, whenever a component in state plus is a left neighbor of a component in state minus, this components on the state plus disappear with probability α.

$$\mathsf{M}_\alpha : \oplus\ominus \to \ominus$$

Variable-Length Analog of Stavskaya Process

We define the interaction operator VS as this composition:

$$\mathsf{VS} = \mathsf{B}_\beta \mathsf{M}_\alpha .$$

An illustration

$$\cdots \qquad \ominus\ominus\ominus\ominus\ominus\ominus \qquad \cdots \quad t = 0$$

An illustration

$$\cdots \qquad \ominus|\ominus|\ominus|\ominus|\ominus|\ominus \qquad \cdots \qquad t = 0$$

An illustration

$$\cdots \qquad \ominus|\ominus|\ominus|\ominus|\ominus|\ominus \qquad \cdots \qquad t=0$$

$$\Lambda$$

An illustration

$t = 0$

An illustration

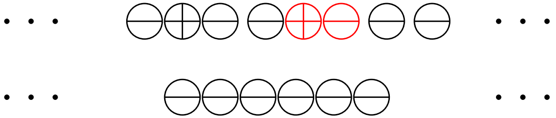
$$\cdots \quad \ominus\oplus\ominus\,|\,\ominus\oplus\ominus\,|\,\ominus\,|\,\ominus \quad \cdots \qquad t = 0$$

An illustration

$$\cdots \qquad \ominus\oplus\ominus\;\ominus\oplus\ominus\;\ominus\;\ominus \qquad \cdots$$

$$\cdots \qquad \ominus\ominus\ominus\ominus\ominus\ominus \qquad \cdots \quad t = 0$$

An illustration



$t = 0$

An illustration

$$\cdots \quad \ominus\oplus\ominus\ominus\ominus\ominus\ominus \quad \cdots \quad t = 1$$

$$\cdots \quad \ominus\ominus\ominus\ominus\ominus\ominus \quad \cdots \quad t = 0$$

Main result

Theorem (Ergodicity)

Lets $\mathcal{M}$ *the set of uniform probability measures on* $\{\oplus, \ominus\}^{\mathbb{Z}}$, $\alpha < 1$ *and* $\beta > 0$. *So,*

$$\forall \quad \mu \in \mathcal{M} : \mu(\mathsf{VS})^t \to \delta_{\oplus} \text{ as } t \to \infty.$$

Computational Study

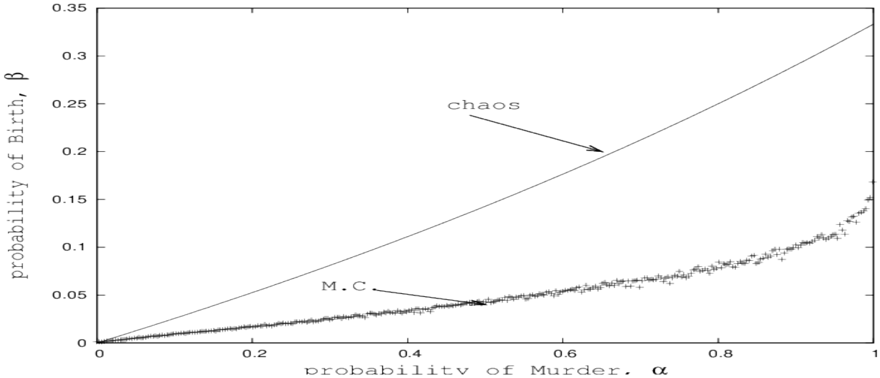


Figure: *This graph shows the curves obtained by Monte Carlo Simulation(M.C.) and meanfield approximation(Chaos). Each curve separates the region where the process seems to be ergodic from that where it seems to be non-ergodic according to each approximation.*

From non-linear VS to a linear process Z

It is well-known that most results in the area of random processes with local interaction speak about linear operators. Unlike that, substitution operators are generally non-linear, which makes them harder to manage. (In a non-linear case, even such an elementary property as transitivity, may take an involved argument[14])

[14] M. A. Raskin. *Toom's Partial Order is Transitive.* Problems of Information Transmission. Vol. 48, pp. 154–172, 2012.

From non-linear VS to a linear process Z

The present case is special: VS, although non-linear, may be reduced to a bi-infinite sequence of random processes, each determined by a linear operator. Let us describe it.

The process Z

Informally speaking, the $(Z_t^{ij})_{t\in\mathbb{Z}_+}$ process can be described by the following scenery:

- **First half of the step:** If there are Z_t^{ij} components pluses between the positions i and j, we will have $Z_t^{ij}+1$ empty words. Every empty word can turn into a plus with probability β or nothing happens with probability $1-\beta$. As each substitution is independent of each other, the number of empty words turned into plus follows a binomial distribution with parameters β and $Z_t^{ij}+1$, i.e.,
$$\Psi_t^{ij} = Z_t^{ij} + X_t,$$
with X_t following a binomial distribution with parameters β and $Z_t^{ij}+1$.

The process Z

- **second half of the step:** for the quantity Ψ_t^{ij} follows: if this quantity is zero, nothing happens; if this quantity is positive, then it is decreased in one unit with probability α, i.e.,

$$Z_{t+1}^{ij} = \Psi_t^{ij} - Y_t \mathbb{I}_{\{\Psi_t^{ij}>0\}}(\Psi_t^{ij})$$

  with Y_t following the Bernoulli distribution with parameter α and $\mathbb{I}_{\{\}}(\cdot)$ is the indicator function.

The **first half of the step** followed by the **second half of the step**, in this order, describes the quantity of plus between two consecutive components in the state zero.

Questions

- Is it possible to obtain a version of Theorem 9 for any initial measure?
- Based on these numerical results, is it true that VS has phases, but not in that simple sense as the classical Stavskaya process?
- Is VS ergodic when $\alpha = 1$ and $\beta > 0$?

Thank you.

.

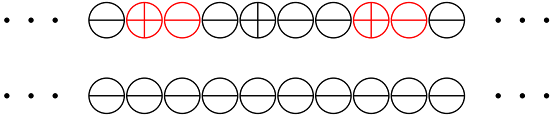
Figure: We are looking for new frontiers!

Some proofs' ideas

Here We introduce a constant length process ν. In this process we sacrifice locality.

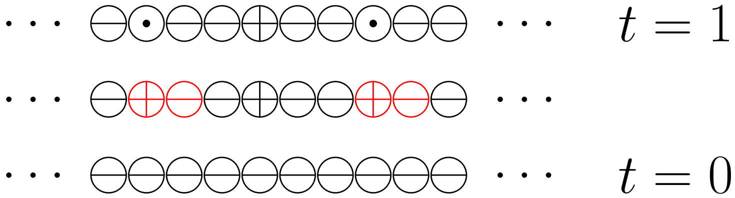
$$\oplus \odot^n \ominus \to \odot^{n+1} \ominus \text{ with probability } \alpha.$$

Some proofs' ideas



$t = 0$

Some proofs' ideas



Some proofs' ideas

To the process ν, we introduce a graphical representation, G, and its respective dual.

Some proofs' ideas

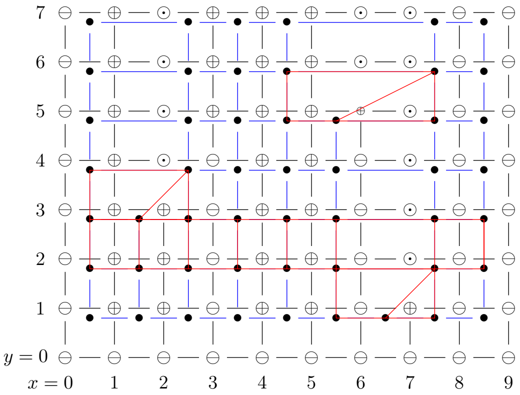


Figure: A fragment of a picture of G and its dual $\overline{G}$. Some figures formed in the dual, which we called rectangles, trapeziums and triangles.

Some proofs' ideas

We define some *types*, which will possibility us to *walk* on G and its respective dual.

Some proofs' ideas

Step in G starting at a $\oplus$-vertex	Type	event	variable
Step west at an even level	1	trivial	none
Step west at an odd level	1'	trivial	none
Step from $(x, 2t+1)$ to $(x, 2t)$ if $F(x,t) =$ move	2	$F(x,t) =$ move	$F(x,t)$
Step from $(x, 2t+1)$ to $(x, 2t)$ if $F(x,t) =$ stay	2'	$F(x,t) =$ stay	$F(x,t)$
Step south from an even to an odd level	2"	trivial	none
Step from $(x, 2t+1)$ to its east neighbor if $M(x,t) =$ fire	3	$M(x,t) =$ fire	$M(x,t)$
Step from $(x, 2t+1)$ to its east neighbor if $M(x,t) =$ stop	4	$M(x,t) =$ stop	$M(x,t)$
Step east at an even level	4'	trivial	none
Step north	5	trivial	none

back

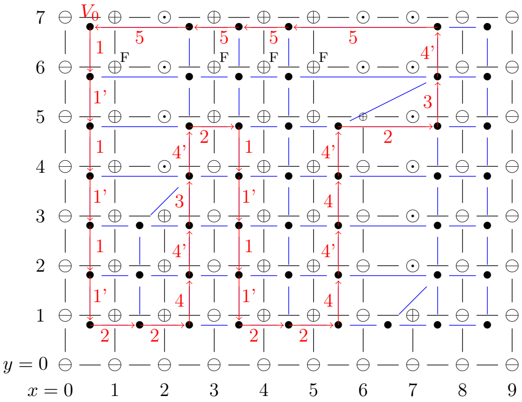


Figure: Illustration of a fragment of the graph G, its dual $\overline{G}$, and a **tour** represented by arrows and their respective types.