On random walks that grow their own domains

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Structure

1. The Model

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2. Random Graph Perspective

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3. Random Walk Perspective

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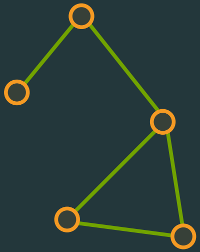

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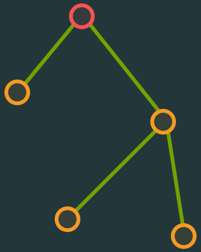

1. A graph is connected if for any pair of vertices u and v, you can 'walk' from u to v and vice-versa;
2. A tree is a connected graph with no cycles
3. A rooted graph is a graph with a distinguishable vertex called the root.

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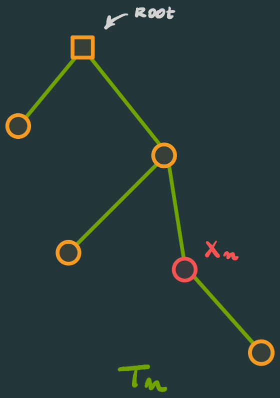

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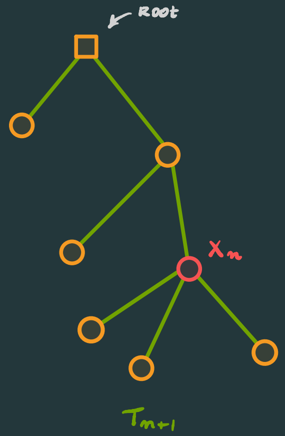

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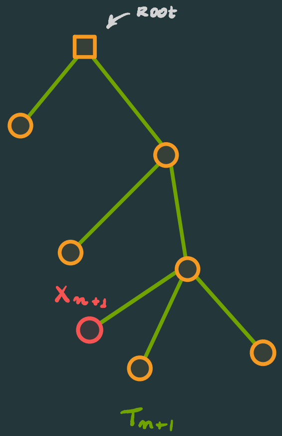

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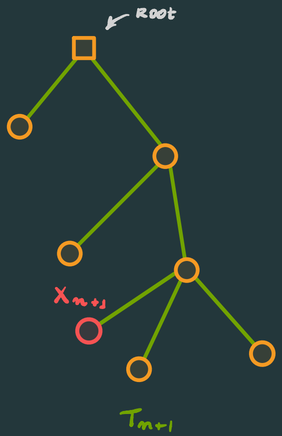

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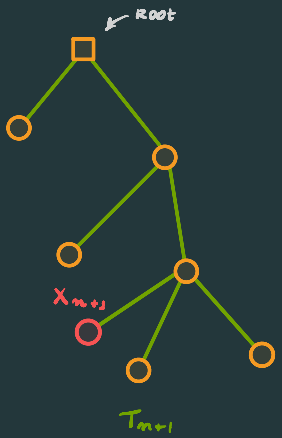

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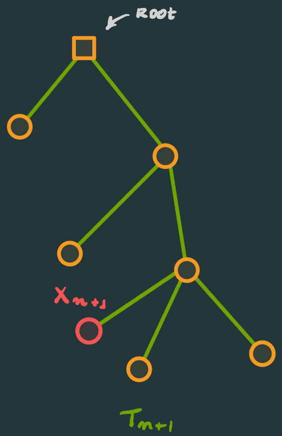
We call this model **Tree Builder Random Walk (TBRW)**.



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2. We can model real phenomenon!

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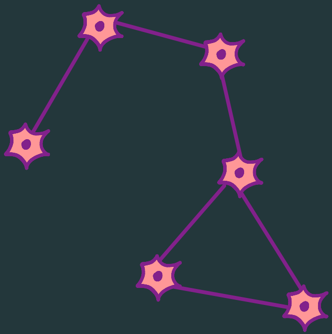

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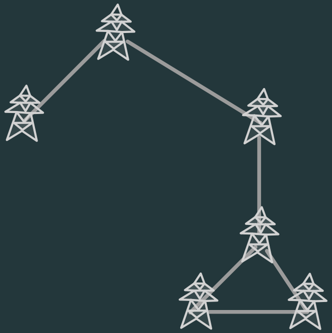

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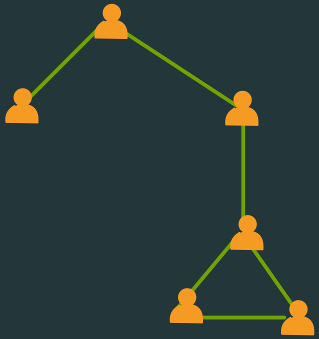

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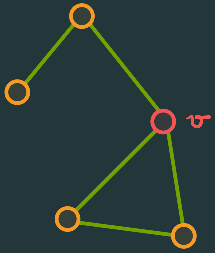

$p_3 = \frac{1}{5}$

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$$\frac{\#\text{ of vertices in G having degree } d}{\#\text{ of vertices in } G} \approx d^{-3}$$

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0 $\qquad$ 1

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Proof.
Left to the reader. =) $\square$

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1. B. Amorim, D. Figueiredo, G. Iacobelli, and G. Neglia. Growing Networks Through Random Walks Without Restarts. Proceedings of the 7th Workshop on Complex Networks CompleNet (2016)

They considered a model that at each $s \in \mathbb{N}$ steps of the walker adds a new vertex at the walker's position. Their simulations suggested for s even, T_n has power-law degree distribution. Whereas for s odd the degree distribution decays exponentially fast on d.

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Observe that $\{X_n\}_{n\in\mathbb{N}}$ is an example *nonmarkovian* random walk. The analysis of $\{X_n\}_{n\in\mathbb{N}}$ imposes some new challenges that we do not see even in the classical theory of RWRE. In this case the environment (domain) is dependent of the walker trajectory.

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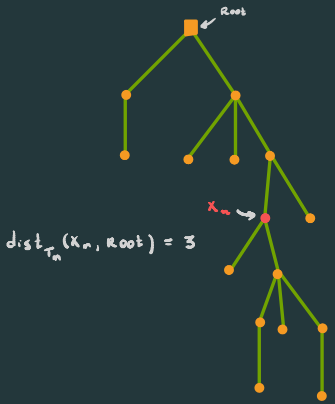

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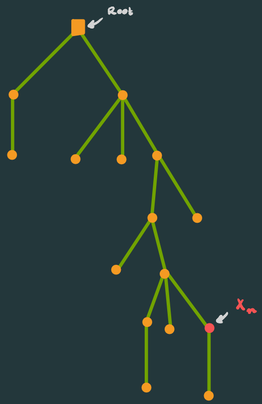

$\mathrm{dist}(X_n, \mathrm{Root}) \geq c \cdot n$

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In words, the above condition means that at each step the walker has probability at least κ of adding at least one new vertex to its position.

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Let $\mathcal{L}_n$ *be a (UE) sequence of distributions over* $\mathbb{N} \cup \{0\}$.

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Proof.
Left to the reader. =) $\square$

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So, $\{\mathcal{L}_n\}_{n \in \mathbb{N}}$ does not satisfy (UE) in the usual sense.

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This explains why T_n showed light tail empirical degree distribution in the simulations.

What about recurrence?

Random Walk Perspective

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3. So the TBRW exhibits all possible regimes for a RW: ballisticity, non-ballistic transience, local traps and recurrence.

General Ideas Behind Recurrence

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expected number of vertices added at time n

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Implies $\mathcal{L}_n(\{1,2,\ldots\}) \searrow 0$.

So, $\{\mathcal{L}_n\}$ is not uniformly elliptic.

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General Ideas Behind Recurrence

(A3) $\underbrace{(1 - q_n)} \cdot M_n^2 \to 0$, as $n \to \infty$, where $M_n := \underbrace{\sum_1^n m_k}$;

Probability of adding at least one vertex at time n.

Expected order of T_n.

$|T_n| \approx M_n$

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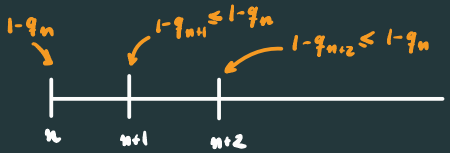

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$$\frac{E\left[\text{Time to add at least one new vertex after time } n\right]}{|T_n|^2} \xrightarrow[n \to \infty]{} 0$$

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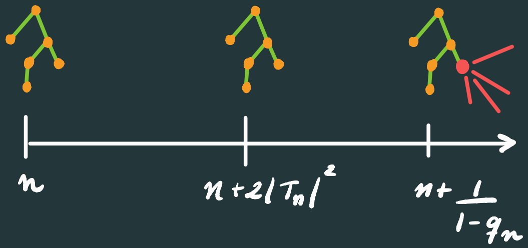

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The key idea is to find a sequence of time intervals $[t_n, t_n + s_n]$ with the following characteristics:

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General Ideas Behind Recurrence

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The right choice is $t_n = n$ and $s_n = n^{2(1-\gamma)+\delta}$

Lemma

Consider a TBRRW where $L_n = \mathrm{Ber}\left(n^{-\gamma}\right)$ *and* $\gamma \in (1/2, 1]$. *Then, for any initial condition* (T, x), *any* $m \in \mathbb{N}$ *(time shift) and* $0 < \delta < 2\gamma - 1$, *it holds that*

$$\mathbb{E}_{T,x;\mathcal{L}^{(m)}}\left[N_{n,n+n^{2(1-\gamma)+\delta}}\right] = o\left(n^{2(1-\gamma)+\delta}\right) .$$

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Theorem (G. Iacobelli, R.R, G. Valle, L. Zuaznabar- Bernoulli - 2020)

General Ideas Behind Ballistic Behavior

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The TBRW

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The TBRW is ballistic whenever both (R) *and* (L) *are satisfied.*

General Ideas Behind Ballistic Behavior

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