Gibbs states and gradient Gibbs states on trees in strong coupling regimes

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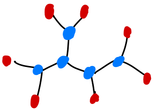
Outline



Setup and background.

$\mathbb{Z}$-valued gradient models and transfer operators Q on trees

Results.

Existence of localized Gibbs measures (GM) for strong coupling, Theorem 1

Existence of delocalized gradient Gibbs measures (GGM), Theorem 2

Existence of spatially inhomogeneous GGM, Theorem 3 + 4

Methods.

Tree-indexed Markov chains, boundary laws, two-layer systems, fixed points, stability analysis for dynamical systems

Related problems.

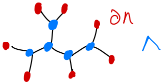
Gibbs specifications γ on trees

(V, E) infinite regular tree of degree d (i.e. $d+1$ nearest neighbors)
V site space
Ω_0 local state space
$\Omega_0 = \{1, \dots, q\}$ finite alphabet $\quad \Omega_0 = \mathbb{Z}$
$\Omega = \Omega_0^V$ infinite volume configurations

Specification γ, on general graphs, for general interactions: a candidate system for conditional probabilities of an infinite-volume Gibbs measure μ (probability measure on Ω) to be defined by **DLR equations**

$$\mu(\gamma_\Lambda(f|\cdot)) = \mu(f) \text{ for all finite volumes } \Lambda \subset V$$

Specification **described by transfer operator** Q



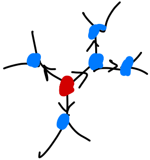
$$\gamma_\Lambda(\omega_\Lambda \mid \omega_{\Lambda^c}) = Z_\Lambda(\omega_{\partial\Lambda})^{-1} \prod_{\substack{\{x,y\}\cap\Lambda\neq\emptyset \\ \textit{nearest neighbors}}} Q(\omega_x, \omega_y).$$

Nearest neighbor specification, spatial Markovianness

Background tree-indexed Markov chains μ - Georgii book Chapter 12

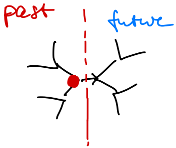
The probability measure μ on $\Omega = \Omega_0^V$ is a **tree-indexed Markov chain** (which is tree-automorphism invariant) iff it allows the iterative construction

1. Sample σ_0 at (arbitrary) root 0 according to single-site marginal of μ
2. Sample σ_w via transition matrix $P(\omega_v, \omega_w)$ from inside to outside



Abstract definition of tree-indexed MC:

$\mu(\sigma_w = \cdot | \mathcal{F}_{\text{past of } (v,w)}) = \mu(\sigma_w = \cdot | \mathcal{F}_v)$ holds for all oriented edges (v, w)

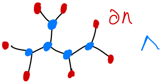


Background Theorem A. μ extremal Gibbs measure $\Rightarrow$ μ tree-indexed MC

Gradient models and transfer operators Q

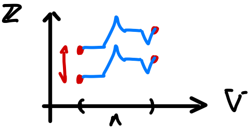
(V, E) infinite regular tree of degree d (i.e. $d+1$ nearest neighbors)
V site space
$\mathbb{Z}$ local state space, height variables
$\Omega = \mathbb{Z}^V$ infinite volume configurations

Gradient specification described by transfer operator $Q : \mathbb{Z} \mapsto (0, \infty)$ depending only on height-differences



$$\gamma_\Lambda(\omega_\Lambda \mid \omega_{\Lambda^c}) = Z_\Lambda(\omega_{\partial\Lambda})^{-1} \prod_{\substack{\{x,y\}\cap\Lambda\neq\emptyset \\ \text{nearest neighbors}}} Q(\omega_x - \omega_y).$$

Note: The kernels γ_Λ are invariant under joint shifts in height-direction



Gibbs measures (GM) vs. Gradient Gibbs measures (GGM)

Assume in the following transfer operator $Q : \mathbb{Z} \mapsto (0, \infty)$ strictly positive

Often $Q(i) = e^{-\beta U(|i|)}$ comes in terms of
gradient interaction potential U, with inverse temperature $\beta \in (0, \infty)$

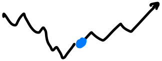
Examples: $U(|i|) = |i|$ SOS model, $U(|i|) = |i|^2$ discrete Gaussian
$U(|i|) = |i|^p$, but in general no monotonicity or convexity is needed for us

Aim: Infinite-volume measures which are consistent with γ, i.e. $\mu\gamma_\Lambda = \mu$

GM: Gibbs measures, probability measures on $\Omega = \mathbb{Z}^V$ absolute heights

GGM: Gradient Gibbs measures, only probability measures on $\Omega/\mathbb{Z}$ increments
(where $\Omega/\mathbb{Z}$ are height configurations modulo a joint height-shift)

Not all GGMs come from GMs (compare two-sided random walk)
Existence not abstractly given, as state-space Ω is non-compact



Purpose of the talk: Outline constructions of
1) Spatially homogeneous measures, localized states
2) (some) spatially inhomogeneous measures (compare Dobrushin-states)

GMs and GGMs on lattices and trees, background

On lattices: $\mathbb{Z}$ or $\mathbb{R}$-valued:

Deuschel, Giacomin, Ioffe, Funaki, Spohn, Sheffield, Kotecky, Luckhaus, Biskup, Lammers-Ott, Dario-Harel-Peled;

Bovier, Orlandi, van Enter, Cotar, K

On trees: Le Ny, Rozikov, Schriever, Henning, K

Background for talk:

1) Henning-K: Coexistence of localized Gibbs measures and delocalized gradient Gibbs measures on trees, AAP 2021

2) Henning-K: Existence of gradient Gibbs measures on regular trees which are not translation invariant, arXiv:2102.11899

3) Henning, K, Le Ny, Rozikov: Gradient Gibbs measures for the SOS model with countable values on a Cayley tree, EJP 2019

Preparations for our existence theorem for localized GMs

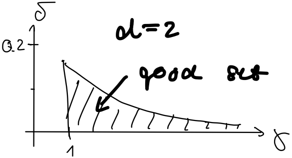
Definition. For any of the spaces

$S = \mathbb{Z}, \quad \mathbb{Z} \setminus \{0\}, \quad \mathbb{Z}_q = \{0, 1, \ldots, q-1\}$ or $\mathbb{Z}_q \setminus \{0\}$,

for any exponent $1 \leq p < \infty$ consider the Banach space

$$l_p(S) := \{x \in \mathbb{R}^S \mid \|x\|_{p,S} := \big(\sum_{j \in S} |x(j)|^p\big)^{\frac{1}{p}} < \infty\}.$$

Lemma. If $\|Q\|_{\frac{d+1}{2},\mathbb{Z}} < \infty$ then the Gibbsian specification kernels γ_Λ are well-defined (i.e. have finite partition functions).

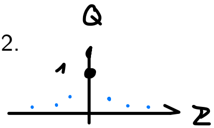


Definition. For any integer $d \geq 2$ define the **good set for interactions**

$$G_d := \{(\gamma, \delta) \in (1, \infty) \times (0, \infty) \mid \text{there exists an } \varepsilon > 0 \text{ such that}$$
$$\delta + \gamma\varepsilon^d \leq \varepsilon \quad \text{and} \quad 2d\gamma\varepsilon^{d-1} + 2d\delta\varepsilon^d < 1\}.$$

Invariance and contractivity

Existence of localized Gibbs measures - gradient models



Theorem 1, Henning-K 2021. Fix any degree of the tree $d \geq 2$.
For any strictly positive transfer operator Q with $Q(0) = 1$
set $\gamma := \|Q\|_{\frac{d+1}{2},\mathbb{Z}}$ and $\delta := \|Q\|_{d+1,\mathbb{Z}\setminus\{0\}}$.

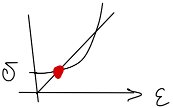
If $(\gamma, \delta) \in G_d$ then there exists a family of distinct tree-automorphism invariant Gibbs measures $(\mu_i)_{i\in\mathbb{Z}}$ which are equivalent under joint translation of the local spin spaces.

↑ localization center

Moreover, the single-site marginal of each μ_i satisfies the following localization bounds

$$\left(\delta \frac{1-\delta\varepsilon(\gamma,\delta)^d}{1+\gamma\varepsilon(\gamma,\delta)^{d-1}}\right)^{d+1} \leq \frac{\mu_i(\sigma_0 \neq i)}{\mu_i(\sigma_0 = i)} \leq \left(\delta \frac{1+\delta\varepsilon(\gamma,\delta)^d}{1-\gamma\varepsilon(\gamma,\delta)^{d-1}}\right)^{d+1}$$

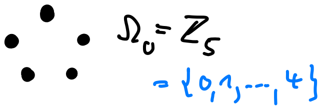
where $\varepsilon(\gamma, \delta)$ denotes the smallest positive solution to the equation

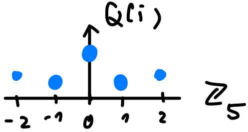


$$\varepsilon = \gamma\varepsilon^d + \delta.$$

low temperature localization

Existence of localized Gibbs measures - clock models





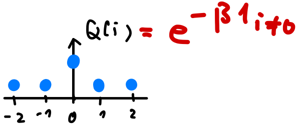
The theorem stays true if the local state space $\mathbb{Z}$ is replaced by the ring $\mathbb{Z}_q = \{0, 1, \dots, q-1\}$ and the transfer operator Q is an even function on $\mathbb{Z}_q$.

Such models are called **clock models** or **discrete rotator models**

and the theorem delivers the existence of ordered phases in this case

low temperature regimes

Specific examples: Potts model $Q(i) = e^{\beta 1_{i=0}}$, discrete Heisenberg model, . . .

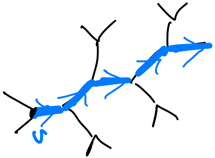


Theorem 2. (Henning, K, 2021). Fix any degree of the tree $d \geq 2$.
Let $Q \in l_1(\mathbb{Z})$ be any strictly positive transfer operator with $Q(0) = 1$.
If $(\|Q\|_{1,\mathbb{Z}}, \|Q\|_{1,\mathbb{Z}\setminus\{0\}}) \in G_d$ then for any $q \geq 2$ there exist tree automorphism invariant GGMs coming from q-periodic boundary law solutions which are not equal to the free state.

i.i.d. increments $\sim \frac{Q(i)}{\|Q\|_1}$

↳ to be explained

Delocalization: $\nu(W_n = k) \overset{n\to\infty}{\to} 0$ along any path of length n and any $k \in \mathbb{Z}$.



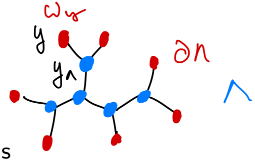
Note: Coexistence of GMs & GGMs possible!

Zachary's Theorem (AoP 1983)

Background Theorem B. There is a one-to-one relation between

- Gibbs measures $\mu \in \mathcal{M}_1(\Omega_0^V)$ which are also tree-indexed Markov chains and
- normalizable **boundary laws** $(\lambda_{xy})_{(x,y)\in\vec{L}}$, where $\vec{L}$ denote oriented edges,

where each $\lambda_{xy} \in (0,\infty)^{\Omega_0}/(0,\infty)$ is a positive measure on local state space (modulo constants) satisfying *consistency* and *normalizability*



The Gibbs measure is described via finite-volume marginals

$$\mu(\omega_{\Lambda\cup\partial\Lambda}) = (Z_\Lambda)^{-1} \prod_{y\in\partial\Lambda} \lambda_{yy_\Lambda}(\omega_y) \prod_{b\cap\Lambda\neq\emptyset} Q(\omega_b),$$

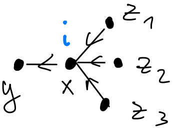
The Markov chain transition operator is

$$P_{xy}(\omega_x,\omega_y) = \frac{Q(\omega_x,\omega_y)\lambda_{yx}(\omega_y)}{\sum_j Q(\omega_x,j)\lambda_{yx}(j)}$$

Boundary Laws - consistency and normalizability

Definition. (Normalizable) boundary laws $(\lambda_{xy})_{(x,y)\in\vec{L}} \in (0,\infty)^{\vec{L}}$ where $\lambda_{xy} \in (0,\infty)^{\Omega_0}/(0,\infty)$ local state space must satisfy the two **defining properties**

$$\lambda_{xy}(i) = c_{xy} \prod_{z\in\partial\{x\}\setminus y} \sum_j Q(i,j)\lambda_{zx}(j) \qquad \text{at any oriented edge } (xy)$$



$$\sum_i \prod_{z\in\partial\{x\}} \sum_j Q(i,j)\lambda_{zx}(j) < \infty \qquad \text{at any site } x$$

necessary to have single-site marginals which are prob. measures

Homogeneous localized boundary law solutions - fixed point method in l^{d+1}

Since a tree-automorphism invariant boundary law $\lambda = (\lambda(i))_{i \in \mathbb{Z}}$ is normalizable if and only if $\lambda \in l_{\frac{d+1}{d}}(\mathbb{Z})$, the family x pointwisely given by $x(i) := \lambda(i)^{\frac{1}{d}}$ corresponds to a normalizable boundary law if and only if $x \in l_{d+1}(\mathbb{Z})$.

Homogeneous boundary laws can be described as fixed points to the operator

$$T : l_{d+1}(\mathbb{Z} \setminus \{0\}) \to l_{d+1}(\mathbb{Z} \setminus \{0\})$$

with

$$T(x)(i) := \frac{Q(i) + \sum_{j \in \mathbb{Z} \setminus \{0\}} Q(i-j)\,|x(j)|^d}{1 + \sum_{j \in \mathbb{Z} \setminus \{0\}} Q(j)\,|x(j)|^d}$$

Important tool: Young convolution inequality in $l_p(\mathbb{Z})$

$$\|u * v\|_r \leq \|u\|_p \, \|v\|_q, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

which gives e.g.

$$\|T(x)\|_{d+1, \mathbb{Z} \setminus \{0\}} \leq \|Q\|_{d+1, \mathbb{Z} \setminus \{0\}} + \|Q\|_{\frac{d+1}{2}, \mathbb{Z}} \underbrace{\|x\|^d_{d+1, \mathbb{Z} \setminus \{0\}}}_{\leq \varepsilon}. \qquad \overset{!}{\leq} \varepsilon$$

invariance and also contractivity of T on a suitable $d+1$ ball

From q-spin GMs to $\mathbb{Z}$-valued GGMs

Fix height-period $q \in \{2, 3, 4, \dots\}$.
Given the transfer-operator $Q \in l^1(\mathbb{Z})$ form the associated
Fuzzy transfer operator $Q^q(i) := \sum_{j \in i + q\mathbb{Z}} Q(i)$
where $i \in \mathbb{Z}_q = \{0, 1, \dots, q-1\}$

Background Theorem C. (Henning, K, 2021). Consider any Gibbs measure (possibly nonhomogeneous) μ on $(\mathbb{Z}_q)^V$ for Q^q.
Then there is an associated GGM ν on $\mathbb{Z}^V/\mathbb{Z}$ which is obtained as a hidden Markov model via edge-wise independent resampling:

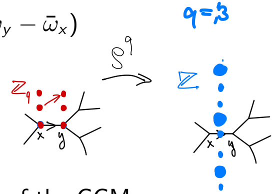
$$\nu(\zeta_\Lambda) = \sum_{\bar{\omega}_\Lambda \in \mathbb{Z}_q^\Lambda} \mu(\bar{\omega}_\Lambda) \prod_{(x,y),\, x,y \in \Lambda} \rho^q(\zeta_{(x,y)} \mid \bar{\omega}_y - \bar{\omega}_x)$$

increments in $\mathbb{Z}$

clock-model

where

$$\rho^q(j \mid \bar{s}) = 1_{j \in \bar{s}} \frac{Q(j)}{Q^q(\bar{s})},$$



In this case, we call the (minimal) q the (height-) period of the GGM ν

Non-homogeneous gradient states and dynamical systems

Theorem 3. (Henning, K, 2021). Consider the gradient specification for the transfer operator $Q \in l^1(\mathbb{Z})$, $Q(i) = Q(-i) > 0$ for all $i \in \mathbb{Z}$, on the regular tree with $d+1$ nearest neighbors.

Then there exists a finite height period $q_0(Q, d)$ such that for all $q \geq q_0(Q, d)$ there are gradient Gibbs measures ν of period q which are not invariant under translations on the tree.

Delocalization, as in homogeneous case: $\nu(W_n = k) \stackrel{n \to \infty}{\to} 0$ along any path of length n and any $k \in \mathbb{Z}$.

Proof idea.

1) Construct non-homogeneous GMs μ on the state space $(\mathbb{Z}_q)^V$ for fuzzy transfer operator Q^q

Find spatially non-homogeneous solutions to Zachary's equation (AoP 1983)

2) Define the gradient state ν via edge-wise resampling of Theorem C

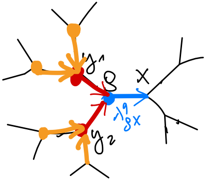
Show that $\mu \mapsto \nu \in \mathcal{M}_1(\mathbb{Z}^V/\mathbb{Z})$ preserves spatial inhomogeneity

Inhomogeneous Boundary law equation on the simplex

Boundary law equation, no symmetries

$$\lambda^q_{xy}(i) = \frac{\prod_{z\in\partial\{x\}\setminus y}\sum_{j\in\mathbb{Z}_q} Q^q_{xz}(i-j)\lambda^q_{zx}(j)}{\|\prod_{z\in\partial\{x\}\setminus y}\sum_{j\in\mathbb{Z}_q} Q^q_{xz}(\cdot-j)\lambda^q_{zx}(j)\|_1}, \quad i\in\mathbb{Z}_q.$$

at any edge $(xy) \in \vec{L}$.



Boundary law equation for solutions with radial symmetry leads to

$$\lambda^q_{\rho x} = H_q(\lambda^q_{y\rho})$$

for an operator

$$H_q : \Delta^q \to \Delta^q$$

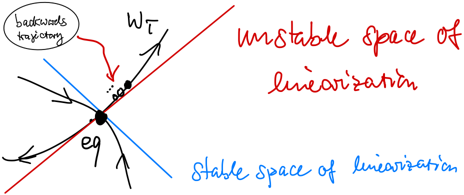
The maps H_q

- For any transfer operator $Q \in l^1(\mathbb{Z})$, for any height-period q, the equidistribution $eq = \frac{1}{q}(1, \dots, 1)$ is a fixed point of H_q

- Idea: construct non-constant infinite backwards-trajectories for H_q to obtain non-homogeneous boundary law solutions.

Chose starting points close to the fixed point eq, but not equal to eq.

- Perform stability analysis around eq:



Problem: non-hyperbolicity (neutral eigenvalues of linearization) at some exceptional parameter values for families of transfer operator Q

Solution via application of τ-unstable manifold theorem (Chaperon 2002) of points which escape from eq at least with rate τ^n with $\tau > 1$.

Existence of nonhomogeneous GGMs via Fourier transform $\hat{Q}$

Theorem 4. Fix any degree $d \geq 2$ of the tree, any height period $q \geq 2$. Suppose that there is a level $\tau > 1$ for which the Fourier transform of the transfer operator

$$\hat{Q} : [-\pi, \pi) \to \mathbb{R}\,; \quad \hat{Q}(k) = \sum_{n \in \mathbb{Z}} Q(n) \cos(nk)$$

satisfies

i) $|\hat{Q}(2\pi \frac{j}{q})| \neq \frac{\tau}{d} \hat{Q}(0)$ for all indices $j \in \{1, \dots, q-1\}$ and

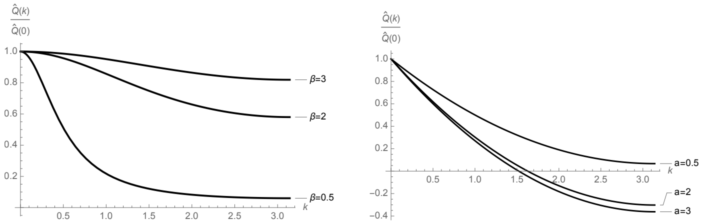
ii) the strict inequality $|\hat{Q}(2\pi \frac{j}{q})| > \frac{\tau}{d} \hat{Q}(0)$ holds for some index $j \in \{1, \dots, q-1\}$.

Then there are gradient Gibbs measures ν of period q which are not spatially homogeneous on the tree.

These states ν are constructed from non-homogeneous radially symmetric boundary law solutions obtained from backwards iteration on the local τ-unstable manifold W_τ around the equidistribution.

Nonhomogeneity of ν is provable for initial values in small neighborhoods

Nonhomogeneous GGMs via $\hat{Q}$: two examples of models



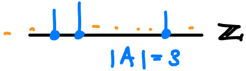
(a) SOS-model (b) Inverse-square model

Figure: Graphs of the function $\hat{Q}(\cdot)/\hat{Q}(0)$ for two models at different parameters.

SOS-model: $Q(i) = e^{-\beta|i|}$ Inverse-square model $Q(i) = 1_{i=0} + \frac{a}{i^2} 1_{i\neq 0}$

Open problems for Gradient models on trees

- Are there homogeneous low temperature GMs μ_A concentrated on finite subsets $A \subset \mathbb{Z}$, which are not convex combinations of μ_i?



This is suggested by analogy to the Potts model on the tree (Khakimov-K-Rozikov JSP 2014, K-Rozikov RSA 2017), and the degenerate zero-temperature case $Q = 1_0$

- Extremality of infinite volume states? Decomposition of states μ^λ ?

$$\mu^\lambda = \int_{\text{ex } \mathcal{G}(\gamma)} w^\lambda(d\nu)\nu$$

GandolfoMaesRuizShlosman JSP 2020, Glassy states: the free Ising model

- Are there spatially inhomogeneous GMs for gradient models of new types?

- Disordered gradient models? $H(\omega) = \sum_{x \sim y} U(|\omega_x - \omega_y|) - \sum_x \eta_x \omega_x$, η_x iid

DarioHarelPeled arXiv2021: Random Field Random Surfaces (on lattices, continuous fields and discrete Gaussian)

Cotar-K AAP2012, PTRF2015: gradient models on lattices, continuous fields

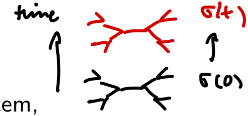
Randomness in geometry AND/OR interaction

Related problems: Time-evolved models on trees, two layer systems

Time-evolved spin models $\mu_t = \mu^\lambda P_t$ on trees show
dynamical Gibbs-non Gibbs transitions
where μ_t non-Gibbs means $\omega \mapsto \mu_t(A|\mathcal{F}_{\Lambda^c})(\omega)$ discontinuous , A cylinder event

Enter-Fernandez-dHollander-Redig CMP02: Lattice-Ising under spin-flip
Iacobelli-Ermolaev-vEnter-K AIHP2012: Tree-Ising under independent spin-flip
Bergmann-Kissel-K, accepted in AIHP: Dynamical Tree-Widom-Rowlinson

Relation to models with **quenched disorder** on trees



Analysis via two-layer systems for $(\sigma(0), \sigma(t))$:
understanding time-zero system conditional on time-t system,
via boundary solutions for spatially inhomogeneous equations

and stability analysis for perturbed maps

model-independent theory? time-evolved gradient models?