Sharp convergence to equilibrium for the SSEP with reservoirs

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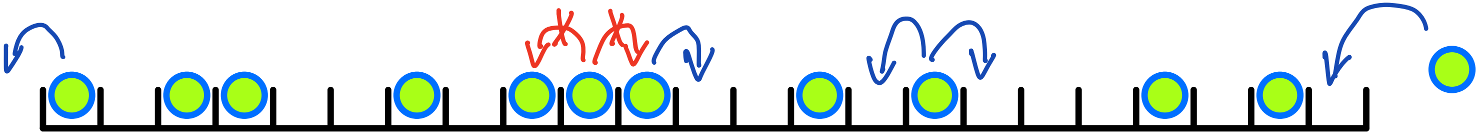
The SSEP with reservoirs



: forbidden

: rate 1

$\Lambda_n := \{1,\dots,n-1\}$; $\Omega_n := \{0,1\}^{\Lambda_n}$

empty site particle

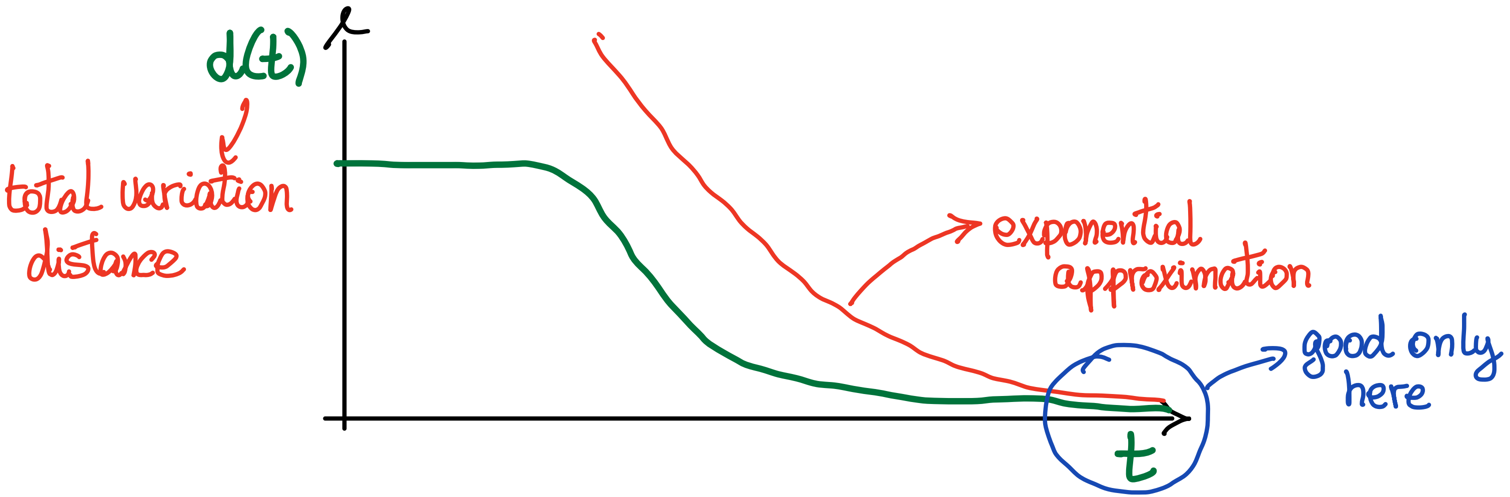
$(\eta_x(t);\ x\in\Lambda_n,\ t\geq 0)$: SSEP with reservoirs

μ: uniform measure in Ω_n

$\hookrightarrow$ invariant for $(\eta(t);\ t\geq 0)$

Markov chain convergence theorem:

The law of $\eta(t)$ converges to μ exponentially fast.

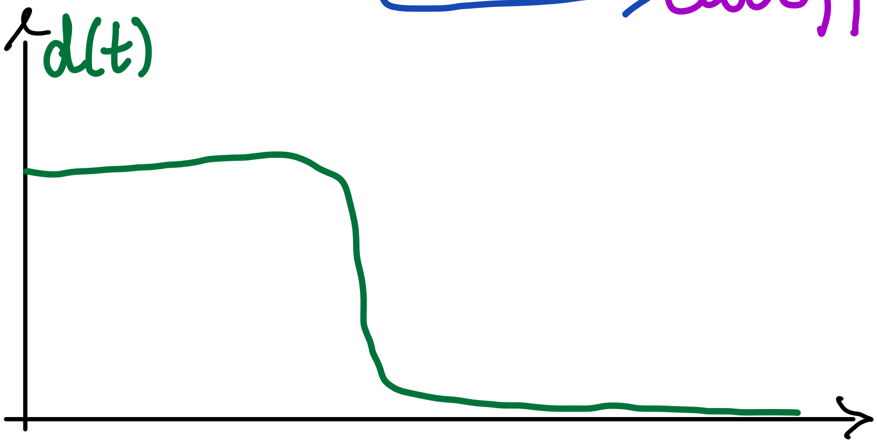


<u>Thm.</u>: (Soley, see also Gantert-Nestoridi-Schmid and Lacoin (AoP '16))

Define $t^n_{\text{mix}}(\varepsilon) := \inf\{t \geq 0 ; d(t) \leq \varepsilon\}$. We have that

$$\lim_{n\to\infty} \frac{t^n_{\text{mix}}(\varepsilon)}{n^2 \log n} = \frac{1}{2\pi^2}.$$

$\longrightarrow$ cutoff phenomenon



Obs: In this theorem, it is used the classical definition

$$d(t) := \sup_{\eta(0)\in\Omega_n} \| \mathrm{law}(\eta(t)) - \mu \|_{TV}$$

total variation

$\hookrightarrow$ worst-case scenario

Natural question: What about "best-case scenario"?

• Hydrodynamics of the SSEP

For $f \in \mathcal{C}([0,1];\mathbb{R})$ and $\eta \in \Omega_n$ define

$$\langle \eta, f \rangle := \frac{1}{n} \sum_{x\in\Lambda_n} \eta_x f\left(\frac{x}{n}\right)$$

Thm. (Hydrodynamic limit)

Let $u_0 : [0,1] \to [0,1]$ be given. Assume that

$$\langle \eta(0), f \rangle \to \int u_0(x) f(x)\, dx \quad \text{in probability}$$

for every $f \in \mathscr{C}([0,1];\mathbb{R})$. Then

$$\langle \eta(tn^2), f \rangle \to \int u(t,x) f(x)\, dx \quad \text{in probability}$$

for every $f \in \mathscr{C}([0,1];\mathbb{R})$, where $(u(t,x);\, t \geq 0,\, x \in [0,1])$ solves

$$\begin{cases} \partial_t u = \Delta u & \text{in } [0,\infty) \times (0,1) \\ u(t,0) = u(t,1) = 1/2 & \text{for } t > 0 \\ u(0,x) = u_0(x) \end{cases}$$

- **Local equilibrium measures:**

$u_0 \in \mathcal{C}([0,1],[0,1])$;

local equilibrium measures

$$\nu^n_{u_0(\cdot)} := \bigotimes_{x \in \Lambda_n} \mathcal{B}ern\left(u_0\left(\frac{x}{n}\right)\right)$$

S^n_t: semigroup associated to the SSEP with reservoirs

$$d_n(t; u_0) := \| \nu^n_{u_0(\cdot)} S^n_{tn^2} - \mu \|_{TV}$$

$$\hat{u}_0(k) := \int \sqrt{2}\,(u_0(x) - 1/2) \sin(\pi k x)\, dx$$

$$\ell_0(u_0) := \inf\{k \in \mathbb{N};\ \hat{u}_0(k) \neq 0\};\quad \gamma_0(u_0) := 2|\hat{u}_0(\ell_0(u_0))|$$

Thm.: (Gonçalves, J., Marinho, Menezes)

Assume that $u_0 \in \mathcal{C}^1([0,1],[\varepsilon_0, 1-\varepsilon_0])$. Then

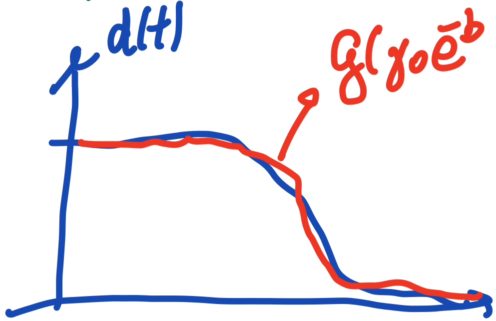
>0

$$\lim_{n\to\infty} d_n\left(\frac{1}{2\pi^2\ell_0^2}\log n + \frac{b}{\pi^2\ell_0^2}; u_0\right) = \mathcal{G}(y_0 e^{-b}),$$

where

$$\mathcal{G}(m) := \|\mathcal{N}(m,1)-\mathcal{N}(0,1)\|_{VT}$$

$$= \sqrt{\frac{2}{\pi}}\int_{m/2}^{\infty} e^{-x^2/2}\,dx.$$



Diaconis, Saloff-Coste AoP

• **Idea of proof:**

Instead of using total variation, we use relative entropy:

$$H(\nu|\mu) := \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu}\, d\mu$$

We use Yau's method from hydrodynamic theory:

$$\mu_t^n := \bigotimes_{x\in\Lambda_n} \mathrm{Bern}\left(u\left(t, \frac{x}{n}\right)\right)$$

↳ solution of HD equation

$$H_n(t) := H\left(\mu_0^n S_{tn^2}^n \,\middle|\, \mu_t^n\right)$$

Thm.: Under the hypotheses of the theorem, there exist $C_0 = C_0(\varepsilon_0, \|\nabla u_0\|_\infty)$, $\delta = \delta(\varepsilon_0, \|\nabla u_0\|_\infty)$ such that

$$H_n(t) \leq C_0 e^{-\delta t}$$

for every $n \in \mathbb{N}$ and every $t \geq 0$.

Classical methods: $H_n(t) \leq C_n e^{-\delta t}$