Stochastic billiards on generalized parabolic domains: recurrence versus transience.

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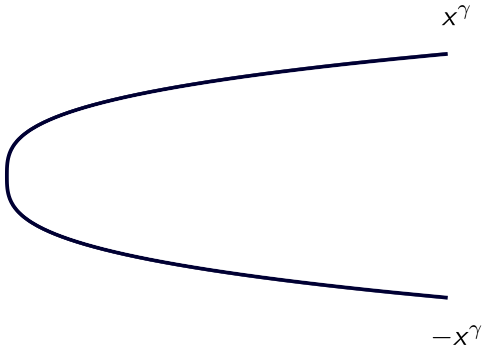
Probability Seminar @ UFRJ

16 Mai, 2022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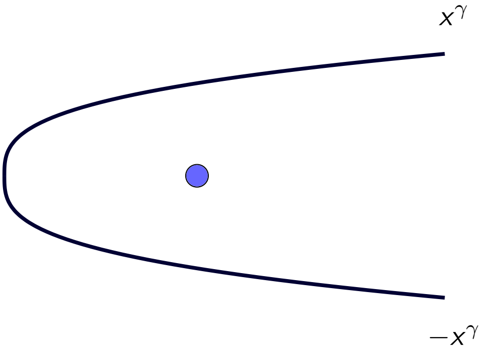
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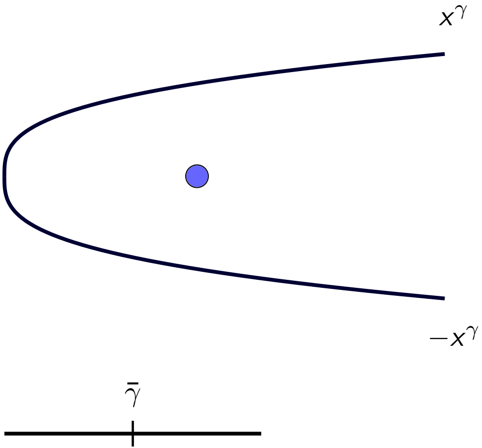
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Outline

The project

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Mikhail Menshikov

Andrew Wade

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08 M. Vachkovskaia, MM, AW Reflecting billiards

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Non-homogeneous random walks

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14 N. Georgiou & AW Half-strip model.

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17	C. Lo & AW	with generalized Lamperti drift.

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	this work	with compact (uncountable) many states.

Background

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Interesting: $d \le 2$ recurrent, $\quad d \ge 3$ transient

Classification heuristics

Supermartingale: $\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] \leq f(X_n)$

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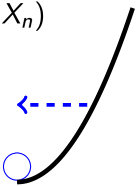
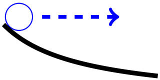


Transient $\Leftrightarrow$



Model

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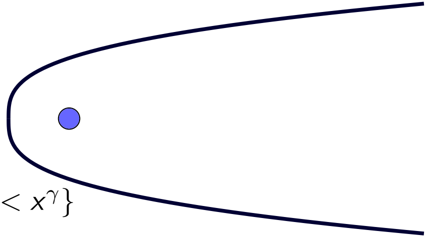

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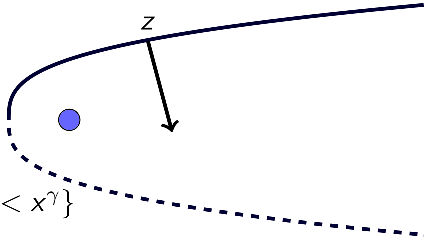

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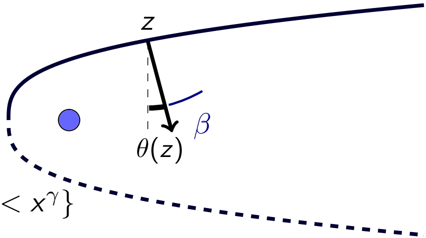

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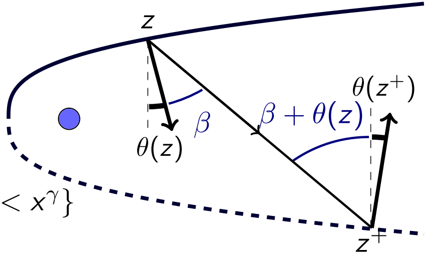

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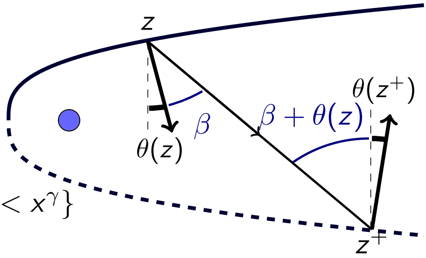

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The construction of the (Markov) process (Z_n, α_n) is complete.

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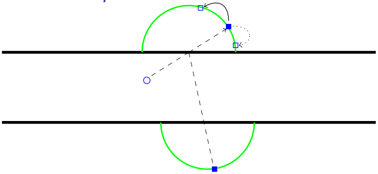

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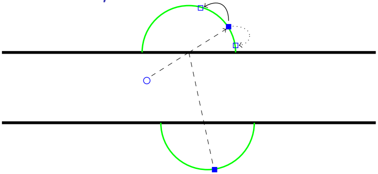

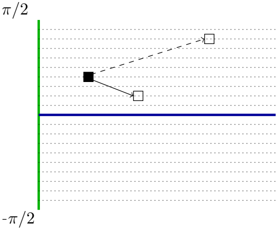

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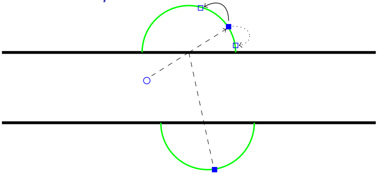

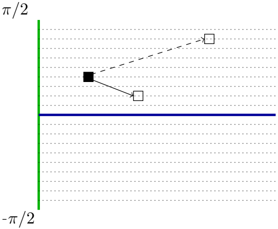

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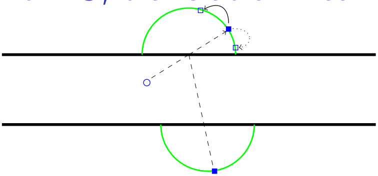

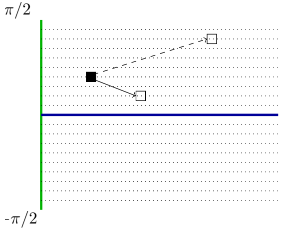

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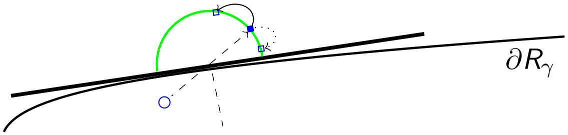

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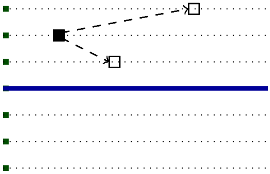

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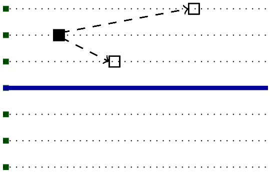

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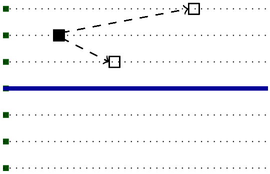

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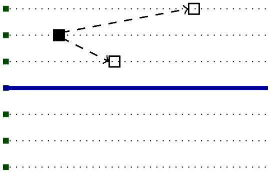

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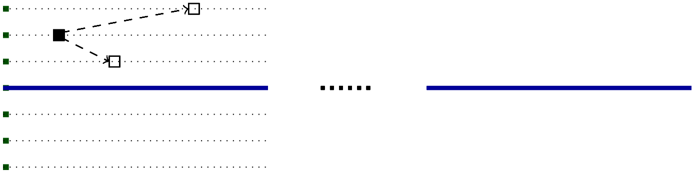

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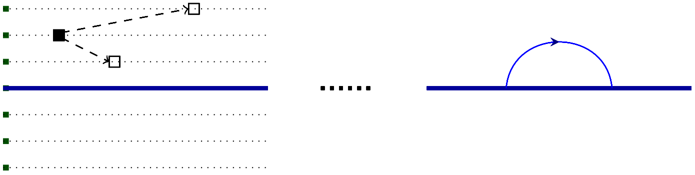

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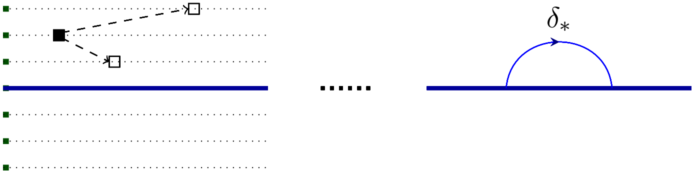

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Non-confinement: $\mathbb{P}(\limsup_n X_n = \infty) = 1$

Lamperti regime

First and second moment conditions

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First and second moment conditions

$$\sup_{\alpha\in S}\left|\mathbb{E}_{x,\alpha}[X_1-x]-\left(d_\alpha+\frac{e_\alpha}{x}\right)\right|=o(x^{-1}),$$

$$\sup_{\alpha\in S}\left|\mathbb{E}_{x,\alpha}[|X_1-x|^2]-s_\alpha^2\right|=o(1),$$

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Lyapunov functions

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with $o_{x,\alpha}$ an error term such that

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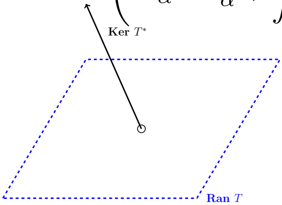

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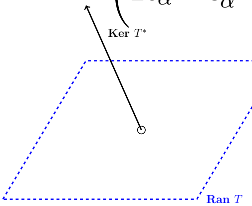

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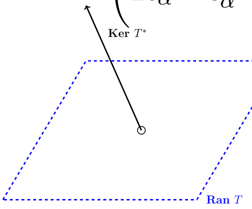

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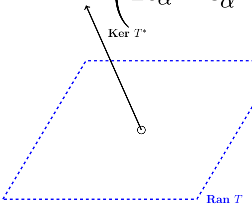

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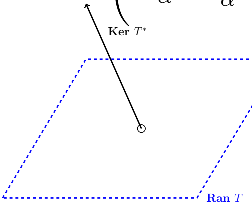

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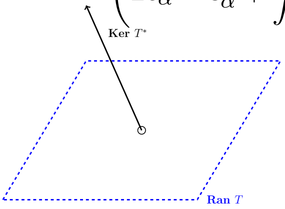

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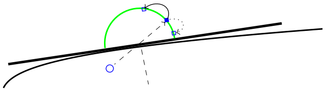
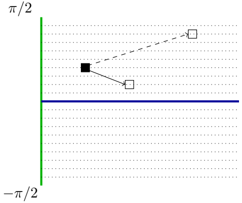
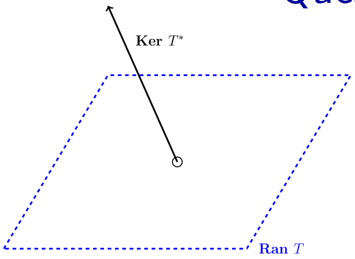
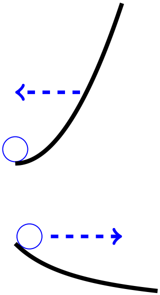




Questions?





The general Result

Generalized Lamperti drift

$$\int d_\alpha \pi(\mathrm{d}\alpha) = 0$$

there exists $\psi_0 \in C_b(S)$ (unique up to translation) with

$$A_1 := \int_S \psi_0(\beta)\varpi(\beta)\tan(\beta)\mathrm{d}\beta \quad A_2 := \int_S \psi_0(\beta)\varpi'(\beta)\mathrm{d}\beta$$

$$c(\gamma) := \gamma(1 + A_1 - A_2 + 2\bar{\rho}_2) - A_1 - \bar{\rho}_2$$

$$c(\gamma) > 0 \quad \text{recurrent}, \qquad c(\gamma) < 0 \quad \text{transient}.$$

$$\psi_0(\alpha) = 2\sum_{n=1}^{\infty} \int_S \mathcal{K}_b(\alpha, \mathrm{d}\beta)\tan\beta.$$

An example

- $\int_S \mathcal{K}_b(\alpha, \mathrm{d}\beta) \tan\beta = \lambda \tan\alpha.$ $\quad \pi(A) = \pi(-A)$

$\gamma < \gamma_c$: recurrent $\quad \gamma > \gamma_c$: transient

In general : $c(\gamma) = \gamma(1 + A_1 - A_2 + 2\bar{\rho}_2) - A_1 - \bar{\rho}_2$

$$\text{if } 1 + A_1 - A_2 + 2\bar{\rho}_2 > 0 : \quad \gamma_g = \frac{A_1 + \bar{\rho}_2}{1 + A_1 - A_2 + 2\bar{\rho}_2}$$

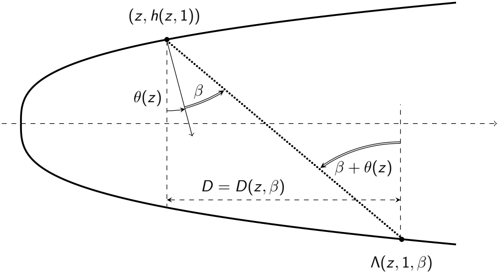
Two chalenges:

- Construct the example
- Show that $\gamma_c = \gamma_g$

Open questions:

- Could $\gamma_c \neq \gamma_g$?
- Is $1 + A_1 - A_2 + 2\bar{\rho}_2 > 0$?
- Is $\gamma_g < 1$?

The displacement



$$\begin{aligned} D &= (z^\gamma + (z+D)^\gamma) \tan(\theta(z) + \beta) \\ &= z^\gamma \left[2 + \gamma z^{-1} D + O(z^{2\gamma-2})\right] \frac{\tan\beta + \tan\theta(z)}{1 - \tan\beta \tan\theta(z)} \\ &= \left(2z^\gamma \tan\beta + 2\gamma z^{2\gamma-1}(1 + \tan^2\beta) + O(z^{3\gamma-2})\right) \\ &\qquad \times \left(1 + \gamma z^{\gamma-1}\tan\beta + O(z^{2\gamma-2})\right) \end{aligned}$$

$$\sup_{\beta\in S_1} \left| D(z,\beta) - \left[2z^\gamma \tan\beta + 2\gamma z^{2\gamma-1} + 4\gamma z^{2\gamma-1}\tan^2\beta\right] \right| = O(z^{3\gamma-2})$$