Asymptotic Behavior of a Low-Temperature of Non-Cascading 2-GREM Hopping Dynamics at Extreme Time Scales

Leonel Zuaznábar
Universidade de São Paulo
joint work with Luiz Renato Fontes (USP) and Susana Frómeta (UFRGS).

The Generalized Random Energy Model (GREM)

Given a natural number N and $p \in (0,1)$ let us define

$$N_1 = \lfloor pN \rfloor \text{ and } N_2 := N - N_1.$$

Consider in $\mathcal{V}_N := \{-1,1\}^N$, a vector $\sigma = \sigma_1\sigma_2$, where

$$\sigma_1 \in \mathcal{V}_{N_1} := \{-1,1\}^{N_1} \text{ and } \sigma_2 \in \mathcal{V}_{N_2} := \{-1,1\}^{N_2}.$$

The Generalized Random Energy Model (GREM)

Given $a \in (0,1)$ and $\sigma = \sigma_1\sigma_2 \in \mathcal{V}_N$, let us consider the Gaussian random variable

$$X_\sigma := \sqrt{a}X^{(1)}_{\sigma_1} + \sqrt{1-a}X^{(2)}_{\sigma_1\sigma_2},$$

where $\{X^{(1)}_{\sigma_1}, X^{(2)}_{\sigma_1\sigma_2} : \sigma \in \mathcal{V}_N\}$ is a family of independent standard Gaussian random variables.

Random Hopping Dynamics

We consider a Markov jump process $\{\sigma^N(t),\ t \geq 0\}$ that evolves in $\mathcal{V}_N$ with transition rates given by

$$\frac{e^{-\beta\sqrt{N}X_\sigma}}{N}\mathbb{1}_{\sigma \overset{1}{\sim} \sigma'} + \frac{e^{-\beta\sqrt{(1-a)N}X_\sigma^{(2)}}}{N}\mathbb{1}_{\sigma \overset{2}{\sim} \sigma'}$$

where, for $i = 1, 2$, we say that $\sigma \overset{i}{\sim} \sigma'$ iff $\sigma \sim \sigma'$ and $\sigma_i \sim \sigma'_i$. Here, $\sigma \sim \sigma'$ indicates that σ and σ' differs in exactly one coordinate.

Previously studied by L.R.Fontes and V. Gayrard

The case $a > p$ was studied by Luiz Renato Fontes and Veronique Gayrard in 2019 ([FG]).

Change of representation

For each N, we relabel the indices $\sigma = (\sigma_1, \sigma_2) \in \mathcal{V}_N$ as $\{\sigma(1), \ldots, \sigma(2^N)\}$ such that

$$X_{\sigma(1)} > X_{\sigma(2)} > \cdots > X_{\sigma(2^N)}.$$

Change of representation

For each N, we relabel the indices $\sigma = (\sigma_1, \sigma_2) \in \mathcal{V}_N$ as $\{\sigma(1), \ldots, \sigma(2^N)\}$ such that

$$X_{\sigma(1)} > X_{\sigma(2)} > \cdots > X_{\sigma(2^N)}.$$

Let us define the function $\phi^N : \mathcal{V}_N \to \mathbb{Z}_+$ as

$$\phi(\sigma) = \phi(\sigma_1\sigma_2) := \min\{i = 1, \ldots, 2^N : \sigma_1(i) = \sigma_1\},$$

and the process X^N as

$$X^N(t) := \phi(\sigma^N(t)), \text{ for all } t \geq 0.$$

Comments

Fine Tuning heuristics

Let $\#_1$ be the number of visits by σ^N to a first level low energy configuration σ_1 before leaving it. This is a geometric random variable with mean

$$1 + \frac{N_2}{N_1} e^{\beta\sqrt{aN} X^{(1)}_{\sigma_1}}$$

Fine Tuning heuristics

Let $\#_2$ be the number of jumps until σ^N finds a second level low energy configuration; it is know that (see Thm 1.6 in [BG])

$$\mathbb{E}[\#_2] \sim 2^{N_2}$$

Fine Tuning heuristics

$$\frac{\mathbb{E}[\#_1]}{\mathbb{E}[\#_2]} \sim \frac{e^{\beta\sqrt{aN}X^{(1)}_{\sigma_1}}}{2^{(1-p)N}} = e^{\beta\sqrt{aN}X^{(1)}_{\sigma_1}-(1-p)N\log 2}.$$

Let us call $\mathcal{P} = \{\xi_i,\, i \geq 1\}$ the Poisson Point Process on $\mathbb{R}$ with intensity measure $e^{-x}dx$, such that

$$\xi_i > \xi_{i+1} \text{ for all } i \geq 1$$

Also let us consider the scaling function for the maximum of 2^N i.i.d. standard Gaussians.

$$u_N(x) = \frac{x}{\beta_* \sqrt{N}} + \beta_* \sqrt{N} - \frac{\log N + \kappa}{2\beta_* \sqrt{N}},$$

for $\kappa = \log \log 2 + \log 4\pi$ and $\beta_* = \sqrt{2 \log 2}$.

Some enviroment result

Theorem 1

For $a < p$ we have that for every $k \geq 1$

$$\left(u_N^{-1}(X_{\sigma(1)}), X^{(1)}_{\sigma_1(1)} - \sqrt{aN}\beta_*; \ldots ; u_N^{-1}(X_{\sigma(k)}), X^{(1)}_{\sigma_1(k)} - \sqrt{aN}\beta_*\right)$$

converges in distribution to $(\xi_1, W_1; \ldots ; \xi_k, W_k)$, where $W_1, \cdots, W_k$ are independent Gaussian variables with mean zero and variance $1 - a$; which are also independent of the process $\mathcal{P}$.

Back to Fine Tuning heuristics

By Theorem 1 we have

$$\frac{\mathbb{E}[\#_1]}{\mathbb{E}[\#_2]} \sim e^{\beta\sqrt{aN}\left(X^{(1)}_{\sigma_1}-\beta_*\sqrt{aN}\right)+\beta_*\left(\beta a-\frac{1-p}{2}\beta_*\right)N}.$$

Back to Fine Tuning heuristics

By Theorem 1 we have

$$\frac{\mathbb{E}[\#_1]}{\mathbb{E}[\#_2]} \sim e^{\beta\sqrt{aN}\left(X^{(1)}_{\sigma_1}-\beta_*\sqrt{aN}\right)+\beta_*\left(\beta a-\frac{1-p}{2}\beta_*\right)N}.$$

Then taking $\beta_{FT} := \frac{(1-p)\beta_*}{2a}$ we have three different scenarios:

Some comments about the proof for Theorem 1

Some notation

Let us denote

$$\gamma_i = e^{\frac{\beta}{\beta_*}\xi_i}$$

and given $L \in \mathbb{R}$ set

$$\mathbb{N}_L = \{i \geq 1 : W_i > L\} \text{ and } \pi_\ell^L = \frac{\gamma_\ell}{\sum_{i \in N_L} \gamma_i} \text{ for } \ell \in \mathbb{N}_L.$$

Theorem 2

For $\beta_* < \beta < \beta_{FT}$, $t > 0, \ell \in \mathbb{N}_L$ and

$$c_N = e^{\beta\left(\beta_* N - \frac{\log N + \kappa}{2\beta_*}\right)} e^{-\beta(\beta_* aN + \sqrt{aN}L)},$$

we have that

$$\lim_{N\to\infty} \frac{1}{c_N t} \int_0^{c_N t} \mathbb{1}_{\{X^N(s)=\ell\}} ds = \pi_\ell^L \text{ (in prob).}$$

Some comments about the scale c_N

Comments on the proof for Theorem 2

Proposition 1

Given $M \geq 1$ such that $\ell \in I_M$, we have for all $t > 0$ that

$$\lim_{N\to\infty} \frac{1}{c_N t} \int_0^{c_N t} \mathbb{1}_{\{X_M^N(s)=\ell\}} ds = \frac{\gamma(\ell)}{\sum_{m=1}^{M} \gamma(i_m)}\,, \text{ in probability.}$$

Proposition 2

Given $t > 0$, let $T_M^{N,out}(t)$ be the time spent by X^N outside $I_M \cup J_M$ up to time t. Then for any $\lambda > 0$ we have

$$\lim_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\left[\frac{1}{c_N t} T_M^{N,out}(c_N t) > \lambda\right] = 0\,.$$

Theorem 3

For $\beta = \beta_{FT}$ we have that $\{X^N(c_N t),\ t \geq 0\}$ converges in distribution as $N \to \infty$ to a K-process on $\mathbb{N}_0 \cup \{\infty\}$ starting from ∞.

Theorem 4

Let us define

$$\bar{c}_N = 2^{-N_2} e^{\beta(\beta_* N - \frac{\log N + \kappa}{2\beta_*})}.$$

Then for $\beta > \beta_{FT}$ we have that $\{X^N(\bar{c}_N t),\ t \geq 0\}$ converges in distribution as $N \to \infty$ to a K-process on $\mathbb{N} \cup \{\infty\}$ starting from ∞.

References

BG Ben Arous, Gérard; Gayrard, Véronique. *Elementary potential theory on the hypercube.* Electronic Journal of Probability 13 (2008): 1726-1807.

BK Bovier, Anton; Irina Kurkova. *Derrida's Generalised Random Energy models 1: models with finitely many hierarchies.* Annales de l'Institut Henri Poincare (B) Probability and Statistics. Vol. 40. No. 4. 2004.

FG Fontes, Luiz Renato; Véronique Gayrard. *Asymptotic behavior and aging of a low temperature cascading 2-GREM dynamics at extreme time scales.* Electronic Journal of Probability 24 (2019): 1-50.

Thanks!