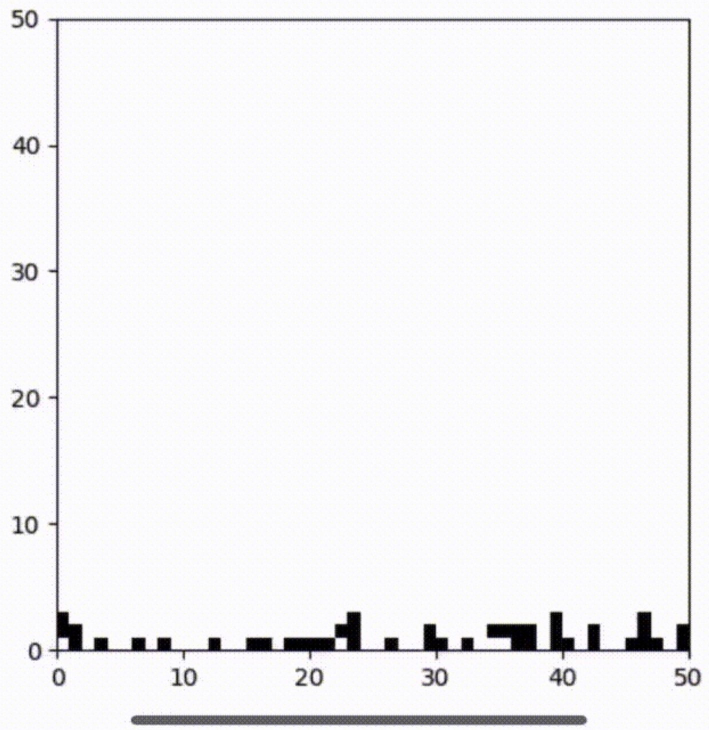
IM-UFRJ Probability Webinar

Scaling Limit for Heavy-tailed Ballistic Deposition with p-sticking

Joint work with Francis Comets & Joseba Dalmau



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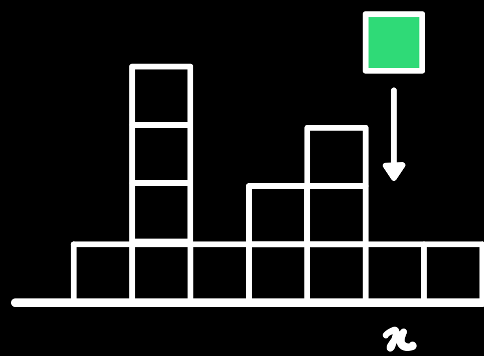
Ballistic Deposition (with p-sticking)

Imagine we have a cluster of blocks growing above the real line which evolves as follows:

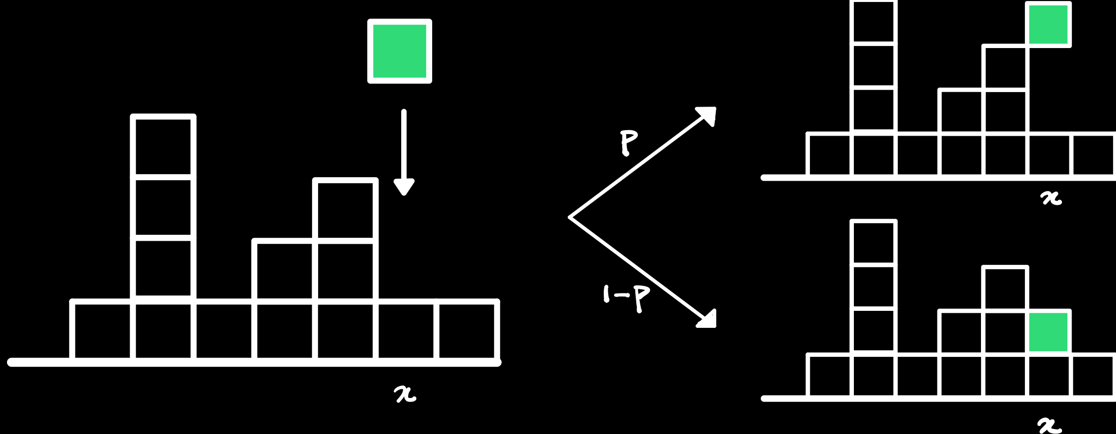
- Above each site $x \in \mathbb{Z}$, 1×1 blocks fall vertically at random with rate 1 (independently for each x).



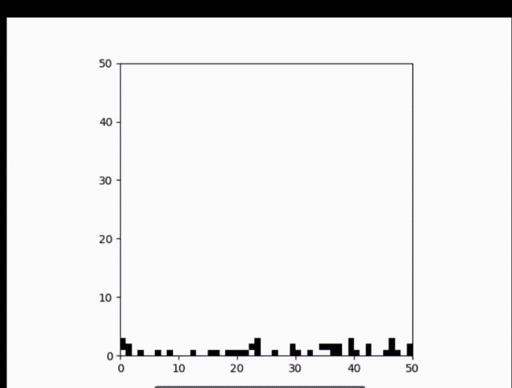
- Whenever a block falls down on x, it has 2 possibilities:

① with prob. $p \in [0,1]$, it sticks to the first point of contact.

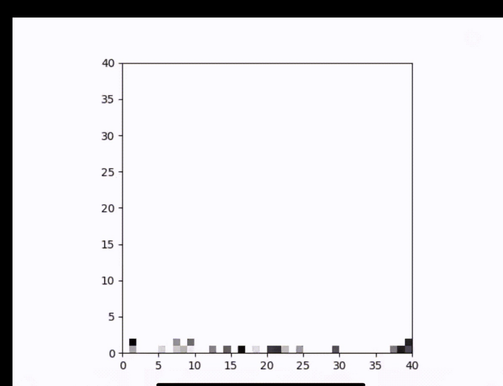
② with prob. $1-p$, it falls straight down and sticks to the last block above x.



p=1 (standard Ballistic Deposition)



p=0 (Random Deposition)



Videos by Tom Weisner

Given $x \in \mathbb{Z}$ and $T > 0$, let us define

$h(x,T) :=$ height at time T of the column of blocks growing above x.

Then:

- For $p = 0$ (RANDOM DEPOSITION):
  - $h(x,T)$ grows linearly in T
  - $h(x,T)$ has Gaussian fluctuations
- For $p = 1$ (BALLISTIC DEPOSITION):
  - $h(x,T)$ grows linearly in T
  - $h(x,T)$ is conjectured to have KPZ fluctuations.

Question How does the system transition from one behavior to the other as p goes from 1 to 0?

Answer: Too hard for us at the moment.

Idea: ask the same question in a different model.

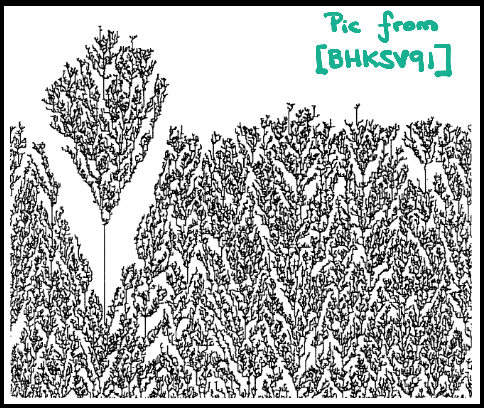
Heavy-tailed Ballistic Deposition (with p-sticking)

The model is the same as BD, with the only difference that the blocks have now random heights, which are i.i.d. with a common CDF F, satisfying:

- F is continuous
- F has a regularly varying (right) tail with index $\alpha \in (0,2)$, i.e.

$$1 - F(t) = t^{-\alpha} L(t)$$

for some slowly varying function L.

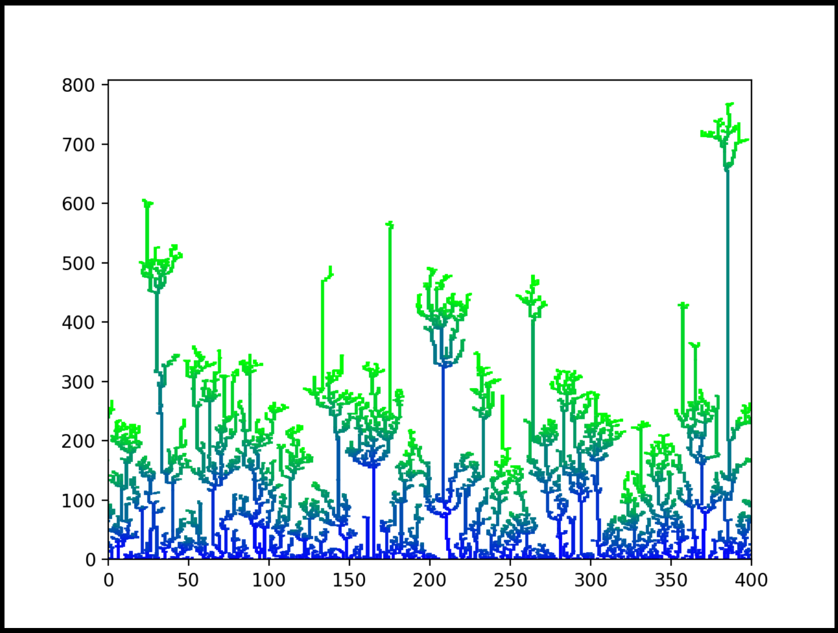


Pic from [BHKSV91]

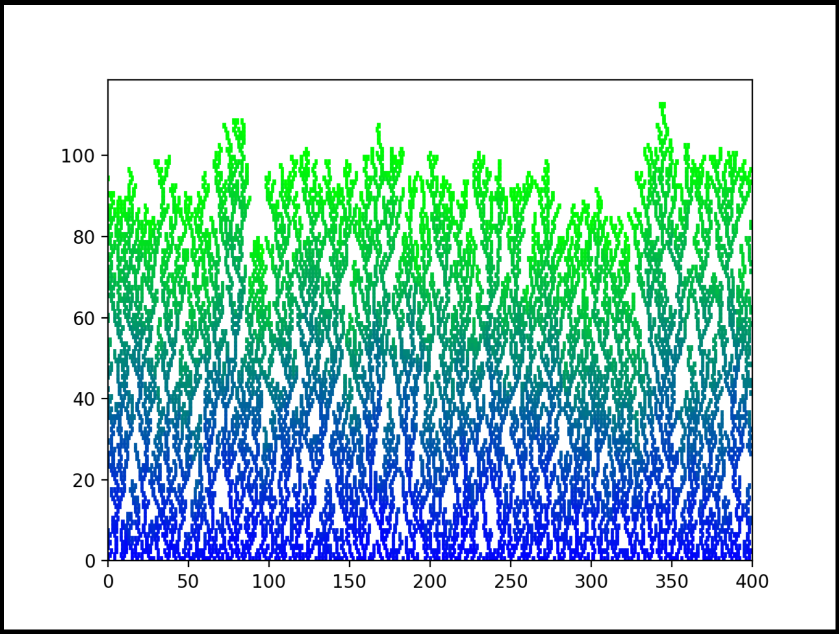
Problem: Understand the rate of growth of the cluster of blocks as $T \to \infty$.

[BHKSV91] = Buldyrev, Havlin, Kertész, Vicsek (Ballistic Deposition with power-law noise)

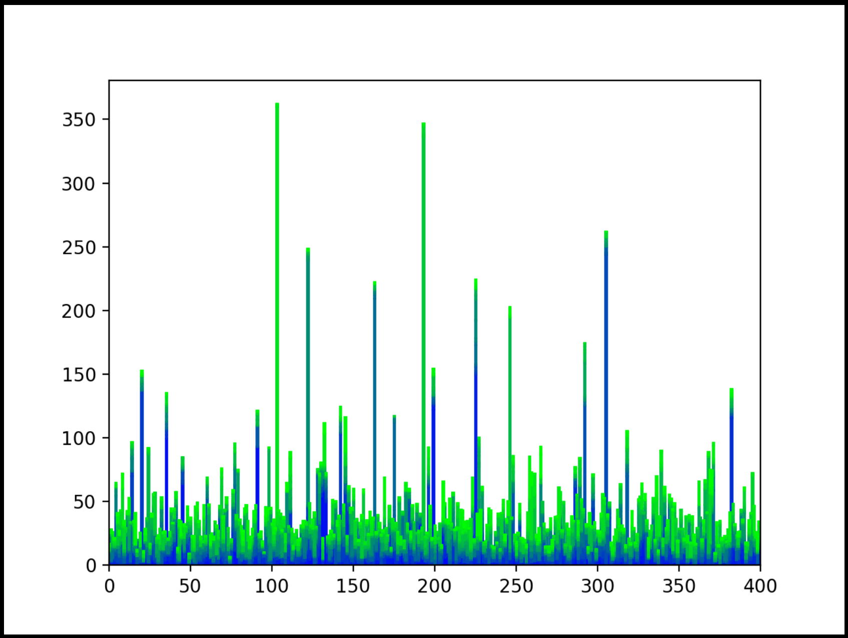
Some pictures ...



Heavy-Tailed Ballistic Deposition



Standard
Ballistic Deposition



Heavy-Tailed
Random Deposition

Main results

Let $h(x,T;p)$ denote the height above $x \in \mathbb{Z}$ of the cluster at time T for the model with p-sticking. Assume that $h(\cdot,0;p) \equiv 0$.

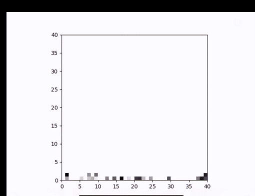
Theorem A There exists a slowly varying function L_F such that for any $p>0$ and $x \in \mathbb{Z}$,

does not depend on p!

$$\frac{h(x,T;p)}{(pT^2)^{\frac{1}{\alpha}} L_F(pT^2)} \xrightarrow[T\to\infty]{d} \mathcal{H}.$$

where $\mathcal{H}$ is characterized via a specific variational formula (later!)

For $p=0$, the behavior is different: the height process $(h(x,T;0))_{T\geq 0}$ is a compound Poisson process with jump CDF F. Thus, by the generalized CLT,



$$\frac{h(x,T;0) - c_T(\alpha)}{T\,\tilde{L}_F(T)} \xrightarrow[T\to\infty]{d} \text{stable law of index } \alpha$$

In particular, $h(x,T;0) \ll T^{2/\alpha} L_F(T^2)$.

Question: What can we say about the phase transition as $p \searrow 0$?

Theorem B Fix $\gamma \in (0,1)$ and, for each $T>0$, set $p_T := T^{-\gamma}$. Then the system $(h(x,T;p_T))_{T\geq 0}$ exhibits a phase transition at $\gamma_c = (2-\alpha)\wedge 1$:

i) $\gamma < \gamma_c \implies \dfrac{h(x,T;p_T)}{(p_T T^2)^{\frac{1}{\alpha}} L_F(p_T T)} \xrightarrow[T\to\infty]{d} \mathcal{H}$

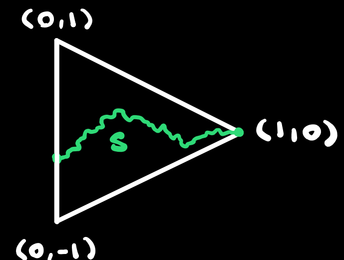
ii) $\gamma > \gamma_c \implies \dfrac{h(x,T;p_T)}{(p_T T^2)^{\frac{1}{\alpha}} L_F(p_T T)} \xrightarrow{\mathbb{P}} \infty$ if $\alpha \in (1,2)$ (and is not tight if $\alpha \in (0,1]$)

The limit $\mathcal{H}$

Let $\mathcal{L} := \{ s : [0,1] \longrightarrow \mathbb{R} \; : \; s(1) = 0 \;,\; |s(t_1) - s(t_2)| \leq |t_1 - t_2|,\ t_1, t_2 \in [0,1] \}$
$= \{ \text{1-Lipschitz paths ending at } 0 \}$

and $\Delta := \{ (t,x) \in \mathbb{R}^2 : |x| \leq 1 - t \}$ = triangle of vertices $(0,1), (0,-1), (1,0)$.

Note that $\text{graph}(s) \subseteq \Delta$ for any $s \in \mathcal{L}$.



Furthermore, let:

- $(U_n)_{n \in \mathbb{N}}$ be i.i.d. random variables uniformly distributed on Δ

- Let $0 < X_1 < X_2 < \dots$ be the sequence of points in a Poisson process with rate 1 independent of $(U_n)_{n \in \mathbb{N}}$ and define $M_n := \frac{1}{(X_n)^{1/4}}$.

  Observe that $(M_n)_{n \in \mathbb{N}}$ is decreasing, i.e. $M_1 > M_2 > \dots > 0$.

- We think of M_n as a (positive) weight positioned at $U_n \in \Delta$.

Then, we define

$$\mathcal{H} := \max \left\{ \sum_{n \in \mathbb{N}} M_n \mathbb{1}_{\{U_n \in \text{graph}(s)\}} : s \in \mathcal{L} \right\}$$

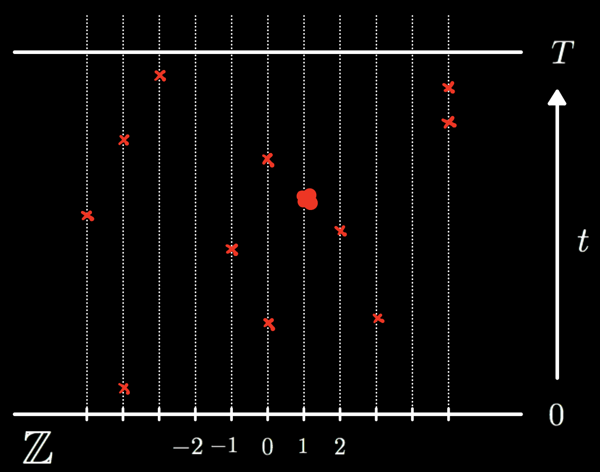
= maximum sum of weights M_n that can be collected by any 1-Lipschitz path ending at 0.

It can be seen that the max is always achieved by an a.s.-unique such path.

Relationship with Last Passage Percolation

For simplicity, let us take $p=1$.

Consider the following "graphical construction" of the process:



- To each $x \in \mathbb{Z}$, assign an independent (marked) Poisson process with rate 1.
- Each $\times$ in the x-th column indicates an occurrence of the corresponding process
- $\times = (t,x)$ means that a block fell down above site x and stuck to the cluster precisely at time t
- Each $\times = (t,x)$ comes with an independent "weight" $\eta(t,x)$ which represents the height of the block

- A (discrete) path $s : [0,T] \longrightarrow \mathbb{Z}$ is called admissible if:

  i) it is cadlag ($\Rightarrow$ piecewise constant)

  ii) all jumps of s are nearest-neighbour (i.e. $\Delta s(t) := s(t) - s(t^-) \in \{-1,0,1\}$ $\forall t$)

  iii) s can only jump if it lands on a $\times$ (i.e. $\Delta s(t) \neq 0 \Rightarrow (t, s(t)) = \times$)

- Define $\mathcal{A}_T(x) := \{ s : [0,T] \longrightarrow \mathbb{Z} :\ s \text{ admissible and } s(T) = x \}$.

Proposition For any $T>0$ and $x \in \mathbb{Z}$, almost surely we have

$$h(x,T;1) = \max\Big\{ h(s(0),0) + \sum_{\Delta s(t)\neq 0} \eta(t,s(t)) \ :\ s \in \mathcal{A}_T(x) \Big\}$$

= maximum sum of weights $\eta(t,x)$ that can be collected by any admissible path ending at x.

Thus, to prove our results, we need to show that this "discrete" optimization problem above is asymptotically equivalent to the "continuous" one defining $\mathcal{H}$ as $T \longrightarrow \infty$.

A few words about the proof

The limit $\mathcal{H}$ is the "same" one obtained by Hambly & Martin (2007) for Heavy-tailed Last Passage Percolation on $\mathbb{N}\times\mathbb{N}$. More precisely, if:

- $(\eta_{i,j})_{i,j\in\mathbb{N}}$ are i.i.d. with common CDF F as before,
- $H_n :=$ maximum sum of weights $\eta_{i,j}$ that can be collected by any increasing path $(0,0)\longrightarrow(n,n)$,

then
$$\frac{H_n}{n^{2/\alpha}L_F(n)} \xrightarrow[n\to\infty]{d} \tilde{\mathcal{H}}.$$

- To prove our results, we will follow the approach by Hambly & Martin (2007).

- **Difficulties**

  1) We don't have a regular lattice like $\mathbb{N}\times\mathbb{N}$, but rather a random "Poissonian lattice". In particular, admissible paths are harder to control than increasing paths.
  2) (discrete) admissible paths $\approx$ 1-Lipschitz paths <u>only</u> as $T\to\infty$.
  3) Hambly & Martin only considered the "$p=1$" case.

- **Main ideas** : How to show that $\dfrac{h(0,T;p)}{(pT^2)^{\alpha}L_F(pT^2)}\approx\mathcal{H}$.

- $\mathcal{E}_T := \{\, x=(t,x) : x \text{ can be "collected" by some } s\in\mathcal{A}_T(0) \,\}$
- Order the weights $\{\eta(t,x) : (t,x)\in\mathcal{E}_T\}$ by size, obtaining a decreasing sequence $M_1^{(T)} > M_2^{(T)} > \dots > M_{k_T}^{(T)}$.
- Let $U_n^{(T)}$, $n=1,\dots,k_T$, denote the position of the weight $M_n^{(T)}$.
- Call $(U_n^{(T)}, M_n^{(T)} : n=1,\dots,k_T)$ the discrete model.
- Call $(U_n, M_n)_{n\in\mathbb{N}}$ the continuous model.

Then, the main steps of the proofs are the following:

1) Show that only the largest weights matter, i.e. for K large,

$$h(0,T;p) \approx \max\left\{ \sum_{n \leq K} M_n^{(T)} \mathbb{1}_{\{U_n^{(T)} \in \text{graph}(s)\}} : s \in \mathcal{A}_T(0) \right\} =: S_K^{(T)}$$

For this, we need good bounds on:

i) $\mathbb{E}(M_n^{(T)})$ for all n,

ii) $\mathbb{E}\left(\max_{s \in \mathcal{A}_T(0)} \left| \{ i \leq n : \tilde{M}_i \text{ is collected by } s \} \right| \right)$

2) Show that the same holds for the continuous model, i.e. for all K large enough,

$$\mathcal{H} \approx \max\left\{ \sum_{n \leq K} M_n \mathbb{1}_{\{U_n \in \text{graph}(s)\}} : s \in \mathcal{L} \right\} =: \mathcal{S}_K$$

3) Construct a coupling between $(U_n^{(T)}, M_n^{(T)})_{n \leq K_T}$ and $(U_n, M_n)_{n \in \mathbb{N}}$ such that:

i) for each $K \in \mathbb{N}$,

$$\frac{1}{(pT^2)^{1/d} L_F(T)} (M_1^{(T)}, \dots, M_K^{(T)}) \xrightarrow[T \to \infty]{\text{c.s.}} (M_1, \dots, M_K)$$

ii) for each i, with high probability as $T \to \infty$,

The optimal path in the discrete model collects the weight $M_i^{(T)}$ $\Longleftrightarrow$ The optimal path in the continuous model collects the weight M_i

(i) + (ii) imply that, for each K, if T is large enough,

$$\frac{1}{(pT^2)^{1/d} L_F(T)} S_K^{(T)} \approx \mathcal{S}_K$$

which, combined with (1) and (2), gives the result.