Small ball probabilities for Gaussian processes

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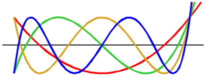
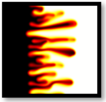
https://yulia-petrova.github.io/





11 July 2022

Probability Seminar - IM-UFRJ

This talk is a small overview, for more details see M. Lifshits “Lectures on Gaussian processes”, 2012, Springer

I would like to thank people who introduced me this topic

Alexander Nazarov

Yakov Nikitin

Mikhail Lifshits

- Alexander I. Nazarov "Variety of fractional Laplacians"
  ICM 2022 speaker: tomorrow — Tuesday, July 12, 14:15 - 15:00

Small ball probabilities: definition

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$$X : (\Omega, \mathbb{P}) \to \mathcal{X} \quad (1)$$

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Actually, it can be formulated as a problem in measure theory. Let P denote the distribution of X, that is a measure in $\mathcal{X}$, given by $P(A) = \mathbb{P}(X \in A)$, and let $U := \{x \in \mathcal{X} : \|x\| \leqslant 1\}$ be the unit ball in $\mathcal{X}$, then we want to study the measure of the small balls:

$$P(\varepsilon U), \quad \text{as} \quad \varepsilon \to 0.$$

Gaussian random vectors

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The distribution of a Gaussian vector is uniquely determined by:

- means of $\{g(X) : g \in \mathcal{X}^*\}$;
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Main example

Wiener process $W(t)$ — a random element in $C[0,1]$ or in $L^2[0,1]$:

- $\mathbb{E}W(t) \equiv 0$;
- $\operatorname{cov}(W(s), W(t)) = \min(s,t)$.

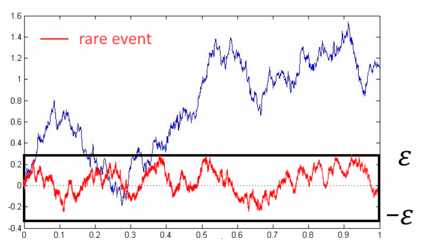
Example

Typical answer:

$$\mathbb{P}\left(\|X\| < \varepsilon\right) \sim D \cdot \varepsilon^{C} \cdot \exp\left(-B\varepsilon^{-A}\right), \qquad \varepsilon \to 0$$

A, B — *logarithmic* asymptotics; A, B, C, D — *exact* asymptotics

Example: $\mathcal{X} = C[0,1]$, $X = W(t)$ — Wiener process



$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |W(t)| < \varepsilon\right) \sim \frac{4}{\pi}\, \exp\left(-\frac{\pi^2}{8}\, \varepsilon^{-2}\right)$$

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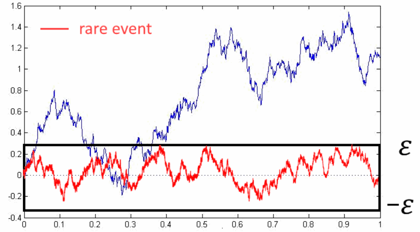
Example: $\mathcal{X} = C[0,1]$, $X = B(t)$ — Brownian bridge



$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |B(t)| < \varepsilon\right) \sim \sqrt{2\pi}\,\varepsilon^{-1}\;\exp\Bigl(-\frac{\pi^2}{8}\,\varepsilon^{-2}\Bigr)$$

Methods

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Exist various methods, among others:

1. spectral method:
   - works for $\mathcal{X}$ being a Hilbert space
   - allows to get exact asymptotics
   - St Petersburg school:
     started by I. Ibragimov, M. Lifshits, Ya. Nikitin, A. Nazarov, and followed by R. Pusev, A. Karol, N. Rastegaev, Yu. Petrova, etc

2. via metric entropy:
   - works for general classes of processes
   - allows to get only logarithmic asymptotics
   - J. Kuelbs, W. Li, W. Linde, T. Dunker, F. Gao, M. Lifshits, F. Aurzada, T. Kuhn, E. Belinsky, R. Blei, W. Salkeld etc

Gaussian processes in Hilbert space

Karhunen-Loeve expansion (KL-expansion):

(K. Karhunen'1947, M. Loève'1948)
Let $\mathcal{X}$ be a separable Hilbert space with orthonormal basis (e_j). Then any Gaussian process X can be represented as

$$X(t) \stackrel{d}{=} \sum_{k=1}^{\infty} e_k \, \xi_k,$$

for ξ_k, $k \in \mathbb{N}$, independent and $\mathcal{N}(0, \sigma_k^2)$-distributed.

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Main idea

All information about the process is in the variances σ_k^2

Hilbert structure $\Longrightarrow$ spectral problem

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(K. Karhunen'1947, M. Loève'1948) Let $\mathcal{X} = L^2[0,1]$. Then

$$X(t) \stackrel{d}{=} \sum_{k=1}^{\infty} u_k(t)\,\sqrt{\mu_k}\,\xi_k$$

- ξ_k, $k \in \mathbb{N}$, — iid standard normal rv
- $u_k(t)$, μ_k — orthonormal eigenfunctions and positive eigenvalues of covariance operator $\mathbb{G}_X$:

$$\mu_k u_k = \mathbb{G}_X u_k \qquad \Longleftrightarrow \qquad \mu_k\,u_k(t) = \int_0^1 G_X(s,t)\,u_k(s)\,ds.$$

Small ball probability problem ($\varepsilon \to 0$):

$$\mathbb{P}(\|X\|_2 < \varepsilon) = \mathbb{P}\left(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\right).$$

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All information about the process is in spectrum of the covariance operator.

What is already known?

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1998 — T. Dunker, M. A. Lifshits, W. Linde (DLL):
rather simple formulas for

$$\mathbb{P}\left(\sum \mu_k \xi_k^2 < \varepsilon^2\right) \qquad \text{when}$$

- μ_k — decays, logarithmically convex
- $\mu_k = k^{-d}, \quad d > 0,$ — polynomial decay
- $\mu_k = A^{-k}, \quad A > 0,$ — exponential decay

Useful fact: Wenbo Li principle

Let $\widehat{\mu}_k \approx \mu_k$ — some approximation.
Question: How the following probabilities are connected

$$\mathbb{P}\left(\sum \mu_k \xi_k^2 < \varepsilon^2\right) \qquad \text{and} \qquad \mathbb{P}\left(\sum \widehat{\mu}_k \xi_k^2 < \varepsilon^2\right)?$$

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Theorem (Wenbo Li principle 1992, Gao et al. 2003)

Let μ_k, $\widehat{\mu}_k$ — two summable sequences. If

$$0 < \prod_{k=1}^{\infty} \frac{\widehat{\mu}_k}{\mu_k} < \infty, \tag{3}$$

then as $\varepsilon \to 0$

$$\mathbb{P}\left(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\right) \sim \mathbb{P}\left(\sum_{k=1}^{\infty} \widehat{\mu}_k \xi_k^2 < \varepsilon^2\right) \cdot \left(\prod \frac{\widehat{\mu}_k}{\mu_k}\right)^{1/2}$$

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1. Consider a spectral problem for the covariance operator $\mathbb{G}_X$

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3. Use DLL theorem for $\widehat{\mu}_k$ and Wenbo Li principle

Example of a general theorem (Nazarov, Nikitin' 2004)

If eigenvalues μ_k have the asymptotics

$$\mu_k = (\vartheta(k + \delta + O(k^{-1})))^{-d},$$

then for the small deviation probabilities

$$\mathbb{P}(\|X\|_2 < \varepsilon) \sim D\varepsilon^{C} \exp(B\varepsilon^{A}), \qquad \varepsilon \to 0,$$

where $A = A(d)$, $B = B(d, \vartheta)$, $C = C(d, \vartheta, \delta)$, $D = D(\{\mu_k\})$:

$$A = -\frac{2}{d-1}, \quad B = -\frac{d-1}{2}\left(\frac{\pi/d}{\vartheta \sin(\pi/d)}\right)^{\frac{d}{d-1}}, \quad C = \frac{2-d-2\delta d}{2(d-1)}$$

Problem statement in my PhD

$X_0(t)$ — Gaussian process:

- $\mathbb{E}X_0(t) \equiv 0$
- $G_0(s,t) = \mathbb{E}X_0(s)X_0(t)$

$\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right)$ is known

$X(t)$ — finite-dimensional perturbation of $X_0(t)$ of rank m :

- $\mathbb{E}X(t) \equiv 0$
- $G(s,t) = \mathbb{E}X(s)X(t)$

$$\boxed{G(s,t) = G_0(s,t) + \vec{\psi}^{\,T}(s) \cdot D \cdot \vec{\psi}(t)}$$

Parameters of perturbation:

- $\vec{\psi}(t) = (\psi_1(t), \dots, \psi_m(t))^T$
- $D \in M_{m\times m}$ — symmetrical

Question:

How $\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right)$ and $\mathbb{P}\left(\|X\|_2 < \varepsilon\right)$ are related to each other?

Problem statement for Durbin processes

- important in statistics
- appear as limiting processes when building goodness-of-fit tests of ω^2-type when parameters are estimated by the sample

Take the sample $x_1, \ldots, x_n \sim F(x, \theta)$.
$\theta = (\theta_1, \ldots, \theta_m)$ — distribution parameters.

parameters are *known*	parameters are *not known*
($\theta = \theta^0$ fixed)	(estimated from sample)
$\Downarrow$	$\Downarrow$
limiting process —	limiting process —
Brownian bridge $B(t)$	perturbation of $B(t)$ of rank m

Problem:

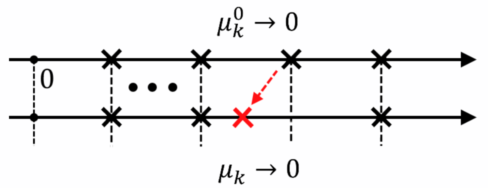
Find exact small ball probability asymptotics for Durbin processes

1-dimensional perturbations: first observation

$$G_X(s,t) = G_0(s,t) + D\psi(s)\psi(t), \qquad D \in \mathbb{R}$$

- $D = 0$ — non-perturbated operator
- $\psi(t)$ — eigenfunction of an integral operator $\mathbb{G}_0$

What will happen if we change D?



Reducing $D \downarrow$

Asymptotically
$\mu_k^0 = \mu_k,\ k \to \infty$

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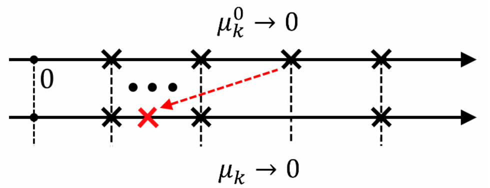

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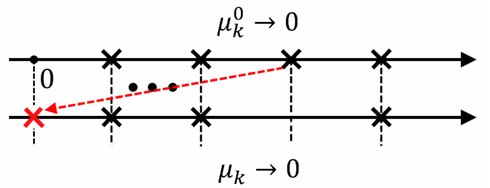

D — critical

Asymptotically
$\mu_k^0 = \mu_{k-1}$, $k \to \infty$

The analogous effect appears in a general situation (when $\psi(t)$ is not necessarily an eigenfunction)

1-dimensional perturbations (A.I. Nazarov'2009)

Let $Q := \langle \mathbb{G}_0^{-1}\psi, \psi \rangle < \infty \quad \Leftrightarrow \quad \psi \in \mathrm{Im}(\mathbb{G}_0^{1/2})$.

There exists a critical value $D_{crit} = -1/Q$ such that:

Non critical case

If $D > D_{crit} = -1/Q$,
then
$$\prod_{k=1}^{\infty} \frac{\mu_k}{\mu_k^0} < \infty$$

Critical case

If $D = D_{crit}$, $\boxed{\psi \in \mathrm{Im}(\mathbb{G}_0)}$,
then
$$\prod_{k=2}^{\infty} \frac{\mu_{k-1}}{\mu_k^0} < \infty$$

- In critical case there is an extra assumption $\psi \in \mathrm{Im}(\mathbb{G}_0)$
- The analogous statement is true for finite-dimensional perturbations (Yu. Petrova'2018)

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Non-critical case

If $D > D_{crit} = -1/Q$,
then as $\varepsilon \to 0$

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) \sim \frac{\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right)}{|1 + QD|}$$

Critical case

If $D = D_{crit}$, $\boxed{\psi \in \mathrm{Im}(\mathbb{G}_0)}$,
then as $\varepsilon \to 0$

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) \sim \frac{\sqrt{Q}}{\|\varphi\|_2} \cdot \sqrt{\frac{2}{\pi}} \cdot$$
$$\cdot \int_0^{\varepsilon^2} \frac{d}{dt}\mathbb{P}(\|X_0\|_2 < t) \cdot \frac{dt}{\sqrt{\varepsilon^2 - t^2}}$$

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Example: Durbin process for Gumbel distribution

Theorem (Yu. Petrova '2017)

For Durbin process $X(t)$ for Gumbel distribution,

$$G(s,t) = \min(s,t) - st - \psi(t)\psi(s), \qquad \psi(t) = C\, t \ln(t) \cdot \ln(-\ln(t))$$

eigenvalue asymptotics is as follows

$$\mu_k^{-1/2} = \pi k + \frac{\pi}{2} + (-1)^k \cdot 2\,\mathrm{arctg}\left(\frac{1}{\ln(\ln(k)) + 1}\right) - \frac{1}{\ln(k)\ln(\ln(k))} + O\left(\frac{1}{\ln(k)(\ln(\ln(k)))^2}\right).$$

Small ball probability asymptotics

$$\mathbb{P}\left\{\|X\|_2 < \varepsilon\right\} \sim C \cdot \ln^{-1}(\ln(\varepsilon^{-1})) \cdot \varepsilon^{-1} \cdot \exp\left(-\frac{1}{8\varepsilon^2}\right)$$

All results:

Non-critical perturbation	Critical perturbation			
	« good »	not « good »	(Durbin processes '2015,'2017)	
1-dimensional: A.I. Nazarov'2009		LOG 2	$\frac{4\sqrt{3+\pi^2}}{3\sqrt{2}\,\pi^{3/2}} \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$	P.'17
		LOG 3	$\frac{4\sqrt{15(3+\pi^2)}}{3\,\pi^{3/2}} \cdot \varepsilon^{-3} \exp\left(-\frac{1}{8\varepsilon^2}\right)$	P.'17
		NOR 1	$C \cdot \varepsilon^{-1} \cdot \ln^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\right) \cdot \exp\left(-\frac{1}{8\varepsilon^2}\right)$	N.& P.'15
		NOR 2	$\frac{2\sqrt{2}}{\pi^{3/2}} \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$	N.& P.'15
finite-dimensional: Yu. Petrova'2018		GUM 1	$\frac{4}{\pi^{3/2}} \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$	P.'17
		GUM 2	$C \cdot \ln^{-1}(\ln(\varepsilon^{-1})) \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$	P.'17
		GAM 1	$\frac{4\alpha_0^{1/2}}{\pi^{3/2}} \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$	P.'17
		GAM 2	$\frac{4d\alpha_0}{\pi^{3/2}} \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$	P.'17

Summing up the fist part

Small ball probability problem consists in finding the asymptotics

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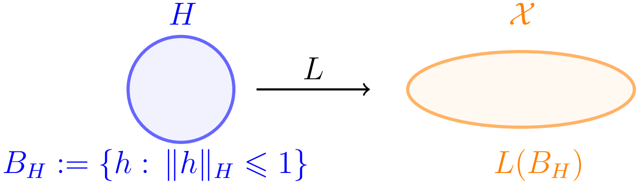
- Hilbert space $\Longrightarrow$ spectral problem
- the whole sequence of eigenvalues μ_k is important (in contrast to large deviations where only the first eigenvalue is sufficient to know)
- very precise asymptotics can be obtained
  ... but it is quite sensitive to any perturbation of the process

Questions? Comments?

Linear operator

Consider an operator $L : H \to \mathcal{X}$ acting between normed spaces.



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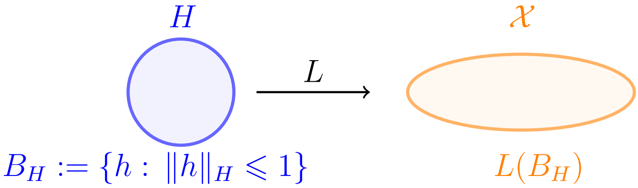

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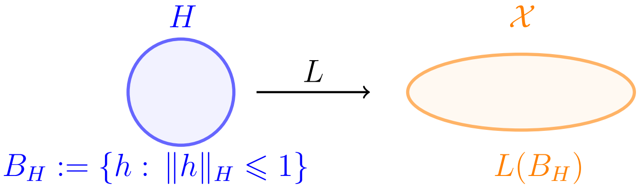
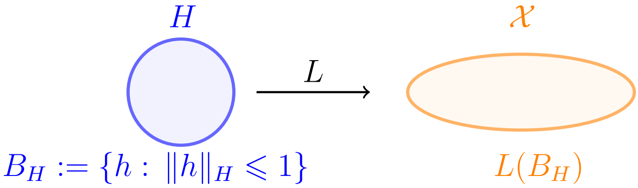


How to measure the "size" of the operator?

- The norm $\|L\|$ (half-diameter of $L(B_H)$) alone is not enough!

Linear operator

Consider an operator $L : H \to \mathcal{X}$ acting between normed spaces.



How to measure the "size" of the operator?

- The norm $\|L\|$ (half-diameter of $L(B_H)$) alone is not enough!
- We can use metric entropy

Covering numbers and entropy

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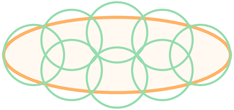

Covering numbers:

$$N_L(\varepsilon) = \inf\left\{n: \exists\{x_j\}_{j\leqslant n}, \{Lh: \|h\|_H \leqslant 1\} \subset \cup_{j=1}^n B_\varepsilon(x_j)\right\}$$

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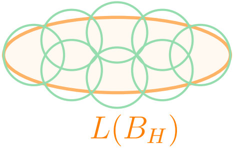

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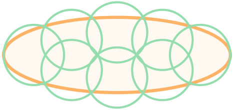

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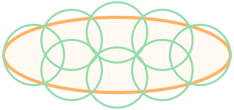

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The main problem in operator language

Find the behavior of covering numbers $N_L(\varepsilon)$, as $\varepsilon \to 0$.

An example: integration operator

1. Let $H = L^2[0,1]$ and $\mathcal{X} = C[0,1]$, and let $L : L^2[0,1] \to C[0,1]$ be an integration operator:

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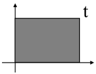
Note that for $\alpha = 1$ this is the simple integration operator. Also there is a semigroup property: $L^\alpha \circ L^\beta = L^{\alpha+\beta}$.

An open problem

Consider multivariate integration operator on $\mathbb{R}^d_+$.
For $t \in \mathbb{R}^d_+$ define a rectangle $[0,t] = \{s : 0 \le s_j \le t_j, 1 \le j \le d\}$.



Let the integration operator L: $L_2([0,1]^d) \to \mathbb{C}([0,1]^d)$ be defined by

$$L(f)(t) := \int_{[0,t]} f(s)\, ds.$$

Problem:

Find the asymptotics for $e_n(L)$, as $n \to \infty$.

It is only known that

- $e_n(L) \approx n^{-1}$, for $d = 1$;
- $e_n(L) \approx n^{-1}(\ln n)^{3/2}$, for $d = 2$;
- $c_1 n^{-1}(\ln n)^{d-1} \le e_n(L) \le c_2 n^{-1}(\ln n)^{d-1/2}$, for general d.

Merging two stories: operators and processes

Any centered Gaussian vector in a separable Banach space $\mathcal{X}$ admits expansion

$$X = \sum_j \xi_j L(e_j), \qquad \text{almost surely,}$$

where ξ_j are iid standard normal rv, and $L : H \to \mathcal{X}$ an appropriate linear operator acting to a $\mathcal{X}$ from a Hilbert space H with basis (e_j).

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Note: the distribution of X doesn't depend on the basis (e_j),

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So we arrive at the expansion

$$W(t) = \xi_0 t + \sqrt{2}\sum_{j=1}^{\infty} \xi_j\, \frac{\sin(\pi j t)}{\pi j}.$$

Metric entropy and Gaussian small deviations

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Relation between $\ln N_L(\varepsilon)$ and $\varphi(\varepsilon)$:

1. polynomial growth:
   Let $\beta \in (0, 2)$. Then

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2. logarithmic growth:
   Let $\beta > 0$, $\gamma \in \mathbb{R}$. Then

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General principles

The following properties are related:

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- small deviation function $\varphi(\varepsilon) := -\ln \mathbb{P}(\|X\| \leqslant \varepsilon)$ is growing slowly when $\varepsilon \to 0$;
- sample paths of a process are rather smooth;
- X has good finite-rank approximations:

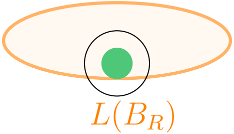
$$X \approx \sum_{j=1}^{n} \xi_j L(e_j), \qquad n \to \infty.$$

How the connection occurs?

We start with an operator $L : H \to \mathcal{X}$. Fix some R, ε. Take the image of the R-ball

$$L(B_R) = \{Lh : \|h\|_H < R\}$$

and construct a pairwise distant points: $h_1, h_2, \ldots$ such that $\|h_i\| < R$ and $\|Lh_i - Lh_j\| > \varepsilon$ for $i \neq j$.



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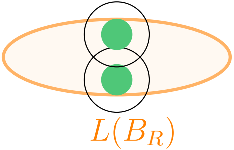

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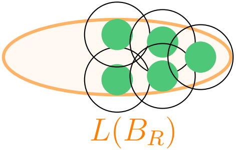

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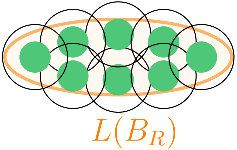

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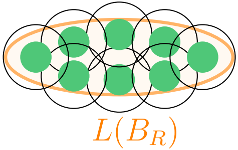

Clearly, we can collect at least $N_{L(B_R)}(\varepsilon)$ points and

$$N_{L(B_R)}(\varepsilon) = N_{L(B_1)}(\varepsilon/R) = N_L(\varepsilon/R).$$

How the connection occurs? Continued

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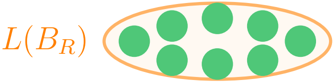

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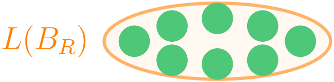

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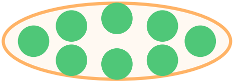

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$$\mathbb{P}(X \in B + Lh) \geqslant \mathbb{P}(X \in B)\exp(-\|h\|_H^2/2).$$

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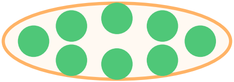
$L(B_R)$ 

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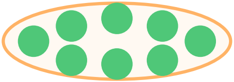

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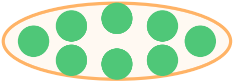

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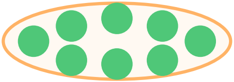
$L(B_R)$ 

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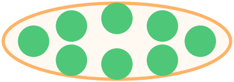

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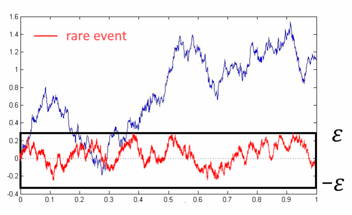
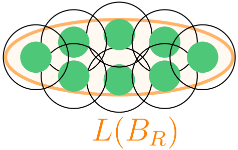
This reads as $\mathbb{P}(\|X\| < \frac{\varepsilon}{2}) \leqslant e^{R^2/2} N_L(\varepsilon/R)^{-1}$. Optimize the RHS in R!

Muito obrigada pela sua atenção!





Questions? Comments?

For any questions: https://yulia-petrova.github.io/

Literature

Own works:

- Nazarov, A. I., Petrova, Yu. P. (2016). The Small Ball Asymptotics in Hilbert Norm for the Kac–Kiefer–Wolfowitz Processes. Theory of Probability & Its Applications, 60(3), 460-480.
- Petrova, Yu. P. (2017). Exact L_2-small ball asymptotics for some Durbin processes. Zapiski Nauchnykh Seminarov POMI, 466, 211-233.
- Petrova, Yu. P. (2021). L_2-small ball asymptotics for a family of finite-dimensional perturbations of Gaussian functions. Zapiski Nauchnykh Seminarov POMI, Nikitin's memorial volume, 501, 236-258.

Overview papers / books :

- Site with all bibliography around small ball probabilities (collected by M. Lifshits): https://airtable.com/shrMG0nNxl9SiGxII/tbl7Xj1mZW2VuYurm
- Li W. V., Shao Q. M. Gaussian processes: inequalities, small ball probabilities and applications. Stochastic Processes: Theory and Methods. North-Holland: Amsterdam, 2001, p. 533-597.
- Lifshits, M. (2012). Lectures on Gaussian processes. Springer, Berlin, Heidelberg.

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Other works:

- Sytaya G. N. On some asymptotic representations of the Gaussian measure in a Hilbert space. Theory of Stochastic Processes, 1974, v. 2, p. 93-104, Kiev (Russian)
- Li. W. V. Comparison results for the lower tail of Gaussian seminorms. J. Theoret. Probab., 1992, v. 5, No 1, p. 1-31.
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- Gao F., Hanning J., Torcaso F. Comparison theorems for small deviations of random series. Electron. J. Probab., 2003, v. 8, No 21, p. 1-17.
- Nazarov A. I., Nikitin Ya.Yu. Exact L_2-small ball behavior of integrated Gaussian processes and spectral asymptotics of boundary value problems. Prob. Theory Related Fields, 2004, v. 129, No 4, p. 469-494.
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Asymptotics of oscillation integrals with slowly varying amplitudes

Let $F(t)$ be a slowly varying function at zero, that means F is of constant sign in the vicinity of 0 and

$$\lim_{x\to 0} \frac{F(\lambda x)}{F(x)} = 1, \qquad \forall \lambda > 0$$

Let $F(1/2) = 0$ and

- $F_{n+1}(x) = xF_n'(x)$ be also slowly-varying functions.

Theorem (A. Nazarov, Yu. Petrova'2016)

The following asymptotic expansion is valid as $\omega \to \infty$:

$$\int\limits_0^{\frac{1}{2}} F(t)\cos(\omega t)\,dt = \frac{1}{\omega}\sum_{k=1}^{N} c_k F_k\Big(\frac{1}{\omega}\Big) + R_N, \tag{4}$$

where $c_k = -\int\limits_0^{\infty} \frac{\sin x}{x}\frac{\ln^{k-1} x}{(k-1)!}\,dx; \qquad |R_N| \le C(F,N)\cdot\frac{\big|F_{N+1}(\frac{1}{\omega})\big|}{\omega}.$

Finite-dimensional perturbations (Yu. Petrova'2018)

$$G_X(s,t) = G_0(s,t) + \vec{\psi}^{\,T}(s) \cdot D \cdot \vec{\psi}(t),$$
$$\vec{\psi}(t) = (\psi_1(t), \ldots, \psi_m(t))^T, \qquad D \in M_{m\times m}$$

Let $\varphi_j(t) = \mathbb{G}_0^{-1}\psi_j(t)$, and

$$Q := \langle \mathbb{G}_0\vec{\varphi}, \vec{\varphi}^{\,T}\rangle < \infty \quad \Leftrightarrow \quad \psi_j \in \mathrm{Im}(\mathbb{G}_0^{1/2})$$

Non-critical case

If $(Q^T D + E_m) > 0,$ then

$$\prod_{k=1}^{\infty} \frac{\mu_k^0}{\mu_k} < +\infty$$

.

Critical case

Если $\boxed{\psi_j \in \mathrm{Im}(\mathbb{G}_0)}$ и $\mathrm{rank}(Q^T D + E_m) = m - s,$ then

$$\prod_{k=s+1}^{\infty} \frac{\mu_k^0}{\mu_{k-s}} < +\infty.$$

- In critical case there is an extra assumption $\psi \in \mathrm{Im}(\mathbb{G}_0)$

Finite-dimensional perturbations (Yu. Petrova'2018)

Теорема (Yu. P. Petrova '2018)

1. *(Non-critical case)* *If* $(Q^T D + E_m) > 0$, *then* $\varepsilon \to 0$

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) \sim \frac{\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right)}{\det(Q^T D + E_m)}.$$

2. *(Critical case)* *If* $(Q^T D + E_m) \equiv 0$, $\boxed{\psi_j \in \mathrm{Im}(\mathbb{G}_0)}$, *then as* $r \to 0$

$$\mathbb{P}\left(\|X\|_2 < \sqrt{r}\right) \sim \sqrt{\frac{\det(Q)}{\det\left(\int_0^1 \vec{\varphi}(t)\, \vec{\varphi}^{\,T}(t)\, dt\right)}} \cdot \left(\sqrt{\frac{2}{\pi}}\right)^m \cdot$$

$$\cdot \int_0^r \int_0^{r_1} \dots \int_0^{r_{m-1}} \frac{d^m}{dr_m^m} \mathbb{P}(\|X_0\|_2 < r_m) \frac{dr_m \dots dr_1}{\sqrt{(r - r_1) \cdot \ldots \cdot (r_{m-1} - r_m)}}.$$

- There exist also partially critical perturbations.