Chasing patterns: diffusion-driven instabilities in networked and high-order systems

Riccardo Muolo, Namur Institute for Complex Systems (naXys), Department of Mathematics, Université de Namur (Belgium)

funded by
My background

BSc Physics and MSc Applied Mathematics
Firenze

PhD Systems Biology (1 year - quit)
Amsterdam

Teaching assistant Mathematics
Namur

Since Jan 2020
PhD Applied Mathematics (exp. 2023)
Namur
Prof. Teo Carletti

Dynamics on networks and beyond group

pattern-formation, random walks, synchronization and more

Possibility of joint Master Thesis and co-supervised PhDs
Outline

Dynamical systems on networks

Turing theory of pattern formation

Turing theory on networks

Effects of non-normality

High-order interactions
Patterns in nature
If there is a link between nodes $i$ and $j$, the adjacency matrix $A_{ij}$ is defined as:

$$A_{ij} = \begin{cases} w_{ij} \in \mathbb{R}^+ & \text{if there is a link between nodes } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$
Networks

Asymmetric (directed)

\[ A_{ij} = \begin{cases} w_{ij} & \in \mathbb{R}^+ \\ 0 & \end{cases} \]

if there is a directed link from node j to node i
Coupled oscillators on networks

\[ \dot{x}_i = f(x_i) \]
Coupled oscillators on networks

\[ \ddot{\vec{x}}_i = \vec{f}(\vec{x}_i) \]

dynamics of \( x_i \)
Coupled oscillators on networks

\[ \dot{x}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j) \]

dynamics of \( x_i \)
Coupled oscillators on networks

\[ \dot{x}_i = f(x_i) + \sigma_1 \sum_j A_{ij}^{(1)} \tilde{g}^{(1)}(x_i, x_j) \]

dynamics of \( x_i \)

pairwise coupling
Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll
Code 6343, Naval Research Laboratory, Washington, D.C. 20375
(Received 7 July 1997)

\[
\dot{\xi} = [1_N \otimes DF + \sigma G \otimes DH] \xi
\]

\[
\dot{\xi}_k = [DF + \sigma \gamma_k DH] \xi_k
\]

Rössler

\[
\begin{aligned}
\dot{x}_i &= -y_i - z_i + \sigma \sum_{j=1}^{N} L_{i,j} x_j \\
\dot{y}_i &= x_i + ay_i \\
\dot{z}_i &= b + z_i (x_i - c)
\end{aligned}
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\end{aligned}
\]
Intermezzo: Turing theory

The Chemical Basis of Morphogenesis

A. M. Turing


\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= f(u, v) + D_u \nabla^2 u(x, t) \\
\frac{\partial v}{\partial t}(x, t) &= g(u, v) + D_v \nabla^2 v(x, t)
\end{align*}
\]

+ boundary conditions
+ domain of the Laplacian
+ ….
Turing theory in a nutshell

reaction-diffusion system of two species (activator $u$ and inhibitor $v$)

homogeneous stable state (fixed point)

inhomogeneous perturbations

exponential instability (diffusion-driven)

patterns $D_v > D_u$
Turing theory in a nutshell

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patterns $D_v > D_u$
Many applications

A dot-stripe Turing model of joint patterning in the tetrapod limb
Jake Cornwall Scones 1 and Tom W. Hiscock 2,3,*

Nanoscale Turing patterns in a bismuth monolayer
Yuki Fuseya, Hiroyasu Katsuno, Kamran Behnia & Aharon Kapitulnik

Linguistic evolution driven by network heterogeneity and the Turing mechanism
Sayat Mimar 1, Mariamo Mussa Juane 2, Jorge Mira 3, Juyong Park 4, Alberto P. Muñuzuri 2, and Gourab Ghoshal 1,*

Turing instability in quantum activator–inhibitor systems
Yuzuru Kato 1,2 & Hiroya Nakao 2
Many applications

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Sayat Mimar,1 Mariamo Mussa Juane,2 Jorge Mira,3 Juyong Park,4 Alberto P. Muñuzuri,2 and Gourab Ghoshal1,∗
1Department of Physics & Astronomy, University of Rochester, Rochester, New York 14607, USA
2Group of Nonlinear Physics, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
3Departamento de Física Aplicada, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
4Graduate School of Culture Technology, Korea Advanced Institute of Science and Technology, Daejeon, 305-701, Korea

OPEN Turing instability in quantum activator–inhibitor systems
Yuzuru Kato1 & Hiroya Nakao2
Extension on networks

\[ L_{ij} = A_{ij} - k_i \delta_{ij} \]

\[
\begin{align*}
\frac{du_i}{dt} &= f(u_i, v_i) + D_u \sum_{j=1}^{\text{nodes}} L_{ij}u_j \\
\frac{dv_i}{dt} &= g(u_i, v_i) + D_v \sum_{j=1}^{\text{nodes}} L_{ij}v_j
\end{align*}
\]
Turing-like instability

PHYSICAL REVIEW E 92, 022818 (2015)

Turing-like instabilities from a limit cycle

Joseph D. Challenger, 1,2 Raffaella Burioni, 3 and Duccio Fanelli 2

the homogeneous stable state is a limit cycle
Coupled oscillators on networks (!)

\[ \dot{x}_i = \tilde{f}(\tilde{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \tilde{g}^{(1)}(\tilde{x}_i, \tilde{x}_j) \]

- dynamics of $x_i$
- chaotic, oscillating, Fixed point, etc...
- pairwise coupling
- in general diffusive like
Coupled oscillators on networks (!)

\[ \dot{x}_i = \underbrace{f(x_i)}_{\text{dynamics of } x_i} + \sum_j A_{ij}^{(1)} g^{(1)}(x_i, x_j) \]

pairwise coupling
in general diffusive like
dynamics of \( x_i \)
chaotic, oscillating,
Fixed point, etc...

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Extensions of Turing theory

PHYSICAL REVIEW E 92, 022818 (2015)

Turing-like instabilities from a limit cycle
Joseph D. Challenger, 1,2 Raffaella Burioni, 3 and Duccio Fanelli 2

The theory of pattern formation on directed networks
Malbor Asllani 1,2, Joseph D. Challenger 2, Francesco Saverio Pavone 2,3,4, Leonardo Sacconi 3,4 & Duccio Fanelli 2

Chaos, Solitons and Fractals 134 (2020) 109707

Generalized patterns from local and non local reactions
Giulia Cencetti a, Federico Battistoni b, Timoteo Carletti c, Duccio Fanelli d,∗
Directed Networks
A network is non-normal when $A^*A \neq AA^*$.

$\beta$-pseudospectrum $\lambda_{\beta}(A) \rightarrow \lambda(A + B) \|B\| < \beta$
Effects on the dynamics

$\omega$ largest eigenvalue of the hermitian (symmetric) part

Linear dynamics

$$A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$$

$\omega(A_2) = 3.52$

$\alpha(A_1) = -1$

$\alpha(A_2) = -1$
Effects on the dynamics

Linear dynamics

Non-linear dynamics

\[ \mathbf{A}_1 = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \]

\[ \mathbf{A}_2 = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix} \]

\[ \omega(\mathbf{A}_2) = 3.52 \]

\[ \alpha(\mathbf{A}_2) = -1 \]

\[ \alpha(\mathbf{A}_1) = -1 \]
Patterns of Non-normality in networked systems

Riccardo Muolo, Malbor Asilani, Duccio Fanelli, Philip K. Maini, Timoteo Carletti
Synchronization and non-normality

Entropi 2021, 23, 36.

Article
Synchronization Dynamics in Non-Normal Networks: The Trade-Off for Optimality

Riccardo Muolo 1,*, Timoteo Carletti 1,*, James P. Gleeson 2, and Malbor Assani 1,2,†

Master Stability Function

Complex MSF

Lyapunov exponent

0 1 2

0 -0.5 -1

0 1 2

0 -0.5 -1
De-synchronization

MSF vs non-linear behavior

basin of attraction

std

0 0.2 0.4 0.6 0.8 1

Log(\beta)

-3 -2 -1 0

Log(\varepsilon)

0 0.5 1

-1.5 -1 -0.5 0

non-normal

symmetric

\(\varepsilon = 0.01\)

\(\varepsilon = 0.05\)

\(\varepsilon = 0.1\)
Debate on optimal networks

Comment on “Synchronization dynamics in non-normal networks: the trade-off for optimality”

Takashi Nishikawa,¹,²,* Adilson E. Motter,¹,² and Louis M. Pecora³

Reply to Comment on “Synchronization dynamics in non-normal networks: the trade-off for optimality”

Riccardo Muolo¹, Timoteo Carletti¹, James P. Gleeson², Malbor Asllani³

Comment on “Synchronization dynamics in non-normal networks: the trade-off for optimality”

Francesco Sorrentino and Chad Nathe

Non-normality, optimality and synchronization

Jeremie Fish⁴
Erik M. Boiti
High-order Structures

a  Network

b  Hypergraph

c  Simplicial complex

Battiston et al., Nat. Phys., 2021
High-order Structures

adjacency matrix

adjacency tensors

a Network

b Hypergraph

c Simplicial complex

Battiston et al., Nat. Phys., 2021
High-order Structures

adjacency matrix

adjacency tensors

Battiston et al., Nat. Phys., 2021
High-order coupled oscillators

\[ \dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A^{(1)}_{ij} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j) \]

dynamics of \( x_i \)

pairwise coupling
High-order coupled oscillators

\[ \ddot{x}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j) + \sigma_2 \sum_{j,k} A_{ijk}^{(2)} \vec{g}^{(2)}(\vec{x}_i, \vec{x}_j, \vec{x}_k) \]

- Dynamics of \( x_i \)
- Pairwise coupling
High-order coupled oscillators

\[ \dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \mathbf{g}^{(1)}(\mathbf{x}_i, \mathbf{x}_j) + \sigma_2 \sum_{j,k} A_{ijk}^{(2)} \mathbf{g}^{(2)}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k) \]

- **dynamics of \( x_i \)**
- **pairwise coupling**
- **high-order (3-body) coupling**
Dynamics on high-order structures

ARTICLE
https://doi.org/10.1038/s41467-021-21486-9
OPEN
Stability of synchronization in simplicial complexes
L. V. Gambuzza, F. Di Patti, L. Gallo, S. Lepri, M. Romance, R. Criado, M. Frasca, V. Latora & S. Boccaletti

Journal of Physics: Complexity

PAPER
Dynamical systems on hypergraphs
Timoteo Carletti, Duccio Fanelli and Sara Nicoletti
Turing theory in high-order systems

Turing patterns in systems with high-order interactions

Riccardo Muolo, Luca Gallo, Vito Latora, Mattia Frasca, Timoteo Carletti

\[ \dot{u}_i = f_1(u_i, v_i) + \sigma_1 D_u^{(1)} \sum_{j_1=1}^{N} A_{i j_1}^{(1)} (h_1^{(1)}(u_{j_1}) - h_1^{(1)}(u_i)) \]

\[ + \sigma_2 D_u^{(2)} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} A_{i j_1 j_2}^{(2)} (h_1^{(2)}(u_{j_1}, u_{j_2}) - h_1^{(2)}(u_i, u_i)) \]

\[ \dot{v}_i = f_2(u_i, v_i) + \sigma_1 D_v^{(1)} \sum_{j_1=1}^{N} A_{i j_1}^{(1)} (h_2^{(1)}(v_{j_1}) - h_2^{(1)}(v_i)) \]

\[ + \sigma_2 D_v^{(2)} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} A_{i j_1 j_2}^{(2)} (h_2^{(2)}(v_{j_1}, v_{j_2}) - h_2^{(2)}(v_i, v_i)) \]
Natural coupling

\[
\frac{d}{dt} \xi = \left( I_N \otimes J_0 + \sigma_1 L^{(1)} \otimes J_H^{(1)} + \sigma_2 L^{(2)} \otimes J_H^{(2)} \right) \xi
\]
Natural coupling

\[ \frac{d}{dt} \vec{\xi} = \left( \mathbb{I}_N \otimes \mathbf{J}_0 + \sigma_1 \mathbf{L}^{(1)} \otimes \mathbf{J}_{H^{(1)}} + \sigma_2 \mathbf{L}^{(2)} \otimes \mathbf{J}_{H^{(2)}} \right) \vec{\xi} \]

same diffusion coefficients for every order

\[ \vec{h}^{(d)}(\vec{x}, \ldots, \vec{x}) = \ldots = \vec{h}^{(2)}(\vec{x}, \vec{x}) = \vec{h}^{(1)}(\vec{x}) \]
Natural coupling

Dispersion law

same diffusion coefficients for every order

\[ \frac{d}{dt} \xi = \left[ I_N \otimes J_0 + \left( \sigma_1 L^{(1)} + \sigma_2 L^{(2)} \right) \otimes J_{H^{(1)}} \right] \xi \]
Regular topologies

\[ L^{(2)} = 2L^{(1)} \]
Regular topologies

\[
\frac{d}{dt} \xi = \left( I_N \otimes J_0 + L^{(1)} \otimes (\sigma_1 J_H^{(1)} + 2\sigma_2 J_H^{(2)}) \right) \xi
\]

\[
L^{(2)} = 2L^{(1)}
\]
Take Home Messages

synchronization of coupled oscillators can be studied in the framework of Turing pattern formation (diffusive coupling)

the topology greatly affects the stability of the synchronized state and a linear stability analysis may fail in its predictions

the theory can be extended in the framework of high-order interactions