Asymptotic Independence via Malliavin-Stein Method

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Motivation and Results

Stochastic Heat Equation (SHE)

$\xi \equiv$ space-time white noise, $W \equiv$ two-sided BM

$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\xi\,,\ Z_0(x) = e^{\sigma W(x)}$

Kardar-Parisi-Zhang Equation (KPZ)

$h_t(x) = \log Z_t(x) \implies \partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi$

Stochastic Burges Equation (SBE)

$u_t(x) = \partial_x h_t(x) \implies \partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \xi$

Forster-Nelson-Stephen 77, Kardar-Parisi-Zhang 86
Bertini-Giacomin 97, Hairer 13

Motivation and Results

Two-Point Correlation Function SBE

$C_t(x) = \mathbb{E}\,[u_0(0)u_t(x)]$, or formally,

$$\mathbb{E}\left[\int_{\mathbb{R}} \phi_1 u_0 dx \int_{\mathbb{R}} \phi_2 u_t dz\right] = \int\int_{\mathbb{R}^2} \phi_1(x)\phi_2(y) C_t(y-x) dxdy\,.$$

Prahöfer-Spohn 04

Asymptotic Independence SBE

How far is the joint measure θ_t of the pair of observables

$$\int_{\mathbb{R}} \phi_1 u_0 dx \text{ and } \int_{\mathbb{R}} \phi_2 u_t dz\,,$$

from the product measure η_t induced by the marginals?

Motivation and Results

Recall $h_0(x) = \sigma W(x)$ and let

- $g_t(x) = \mathrm{Var}\,[h_t(x)]$
- $\psi_1' = \phi_1$, $\psi_1(0) = 0$

Theorem 1

- $C_t(x) = \frac{g_t''(x)}{2}$
- $\mathrm{Wass}(\theta_t, \eta_t) \leq \frac{1}{\sigma\|\phi_1\|_2}\sqrt{\frac{\pi}{2}}\int\int_{\mathbb{R}^2} |\phi_2'(z)||\psi_1(z+x)|\frac{g_t''(x)}{2} dzdx$

Motivation and Results

- SHE $Z_0(x,y) = \delta(x-y) \mapsto Z_t(x,y)$ fundamental solution
- Solution of KPZ $\mapsto h_t(x) = \log\left(\int_{\mathbb{R}} Z_t(x,y) e^{\sigma W(y)} dy\right)$
- End-point density $\mapsto p_t(y) = \frac{Z_t(0,y)e^{\sigma W(y)}}{\int_{\mathbb{R}} Z_t(0,z)e^{\sigma W(z)} dz}$

Alberts-Khanin-Quastel 14

Theorem 2

$$\mathbb{E}\left[p_t(y)\right] = \frac{g_t''(y)}{2\sigma^2}$$

KPZ Universality

- $\epsilon^{1/2} h_{\epsilon^{3/2} t}(\epsilon x) - c_\epsilon t \to \mathfrak{h}_t(x)\,,$ as $\epsilon \searrow 0$
- $g_t(x) = \mathrm{Var}\, h_t(x) \sim t^{2/3} g(x t^{-2/3})$
- $\mathbb{E}\left[p_t(y)\right] \sim t^{-2/3} f\left(y t^{-2/3}\right)$

Related Works

Stationary $\sigma = 1$

- Balázs-Quastel-Seppäläinen 11, Gu-Komorowski 22
- Borodin-Corwin-Ferrari-Vetö 15
- Imamura-Sasamoto 13, Maes-Thiery 17, Le Doussal 17

Narrow Wedge and $\sigma = 0$

- Amir-Corwin-Quastel 10, Corwin-Hammond 16
- Das-Zhu 22

KPZ Fixed Point

- Matetski-Quastel-Remenik 21, Dauvergne-Ortmann-Virag 22
- Quastel-Sarkar 23, Virag 20
- P 22

Part 1: Distance from Independence

- $(X_1, X_2) : \Omega \to \mathbb{R}^2$, $\theta = \mathbb{P}_{X_1,X_2}$ and $\eta = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2}$.
- Wasserstein Distance

$$\mathrm{Wass}(\theta, \eta) = \sup_{\ell \in \mathrm{Lip}_1} \left\{ \left| \int_{\mathbb{R}^2} \ell \, d\theta - \int_{\mathbb{R}^2} \ell \, d\eta \right| \right\} .$$

- $W = (W(x), \, x \in \mathbb{R})$ two-sided standard BM, $\sigma \geq 0$,

$$X_1 = \sigma \int_{\mathbb{R}} \phi_1 \dot{W} \, dx \text{ and } X_2 = X_2(\sigma, W) .$$

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Malliavin Derivative

- $W(\phi) = \int_{\mathbb{R}} \phi \dot{W} dx$.
- For $X = f(W(\phi_1), \dots, W(\phi_d))$

$$DX(u) = \sum_{j=1}^{d} \partial_{x_j} f(W(\phi_1), \dots, W(\phi_d))\phi_j(u) \in \mathbb{L}^2(\Omega \times \mathbb{R}).$$

- $DX_1(u) = D(\sigma W(\phi_1))(u) = \sigma \phi_1(u)$.

Lectures Notes by D. Nualart

Malliavin Derivative

- Chain rule (CR)

$$Df(X_1, X_2) = \partial_{x_1} f(X_1, X_2) DX_1 + \partial_{x_2} f(X_1, X_2) DX_2\,;$$

- Integration by parts (IP)

$$\mathbb{E}\left[W(\phi)X\right] = \mathbb{E}\left[\langle DX\,,\,\phi\rangle\right]\,;$$

- Correlation $\mathbb{E}\left[X_1 X_2\right] = \sigma \mathbb{E}\left[\langle DX_2, \phi_1\rangle\right]$;

Stein's Method - Asymp. Independence

- Let $\mathcal{N} f(x_1, x_2) = \sigma_1^2 \partial_{x_1} f(x_1, x_2) - x_1 f(x_1, x_2)$. Then

$$X_1 \sim N(0, \sigma_1^2) \text{ and } X_1 \perp X_2 \iff \mathbb{E}\left[\mathcal{N} f(X_1, X_2)\right] = 0\,.$$

- For $X_1 = \sigma \int_{\mathbb{R}} \phi_1 \dot{W} dx \sim \mathrm{N}(0, \sigma_1^2)$, with $\sigma_1^2 = \sigma^2 \|\phi_1\|_2^2$, use

$$\mathbb{E}\left[\mathcal{N} f(X_1, X_2)\right]$$

to quantify distance from independence.

Lectures Notes by S. Chatterjee

LEMMA 1

Given a nice function ℓ, there exists a unique bounded solution $f_\ell : \mathbb{R}^2 \to \mathbb{R}$ of the equation

$$\mathcal{N} f(x_1, x_2) = \ell(x_1, x_2) - \mathbb{E}\left[\ell(X_1, x_2)\right] .$$

Furthermore,

- $\|f_\ell\|_\infty \leq \|\partial_{x_1}\ell\|_\infty$;
- $\|\partial_{x_1} f_\ell\|_\infty \leq \frac{1}{\sigma_1}\sqrt{\frac{2}{\pi}}\|\partial_{x_1}\ell\|_\infty$;
- $\|\partial_{x_2} f_\ell\|_\infty \leq \frac{1}{\sigma_1}\sqrt{\frac{\pi}{2}}\|\partial_{x_2}\ell\|_\infty$.

Proof

Let $\ell : \mathbb{R}^2 \to \mathbb{R}$ be a nice function and $Y \sim \mathrm{N}(0, \sigma_1^2)$. Then

$$f_\ell(x_1, x_2) = -\frac{1}{\sigma_1^2} \int_0^1 \frac{1}{2\sqrt{t(1-t)}} \mathbb{E}\left[Y \ell\left(\sqrt{t} x_1 + \sqrt{1-t}\, Y, x_2 \right) \right] dt \,.$$

Use that $\mathbb{E}\,[Yf(Y)] = \sigma_1^2 \mathbb{E}\,[f'(Y)]$.

Stein's Method - Asymp. Independence

If $\theta = \mathbb{P}_{X_1,X_2}$ and $\eta = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2}$

$$\Longrightarrow \int_{\mathbb{R}^2} \ell \, d\theta - \int_{\mathbb{R}^2} \ell \, d\eta = \mathbb{E}\left[\mathcal{N} f_\ell(X_1, X_2)\right].$$

Wasserstein Distance to Independence

$$\mathrm{Wass}(\theta, \eta) \leq \sup_{f_\ell} \left| \mathbb{E}\left[\mathcal{N} f_\ell(X_1, X_2)\right] \right|.$$

Malliavin-Stein Method - Asymp. Indep.

Recall $X_1 = \sigma W(\phi_1) \sim \mathrm{N}(0, \sigma_1^2)$ with $\sigma_1^2 = \sigma^2 \|\phi_1\|_2^2$.

$\mathbb{E}\left[X_1 f(X_1, X_2)\right] = \sigma \mathbb{E}\left[W(\phi_1) f(X_1, X_2)\right]$ equals (IP+CR)

$$\begin{aligned}
&\sigma \mathbb{E}\left[\langle Df(X_1, X_2), \phi_1\rangle\right] \\
&= \sigma \mathbb{E}\left[\partial_{x_1} f(X_1, X_2)\langle DX_1, \phi_1\rangle\right] + \sigma \mathbb{E}\left[\partial_{x_2} f(X_1, X_2)\langle DX_2, \phi_1\rangle\right] \\
&\qquad = \sigma_1^2 \mathbb{E}\left[\partial_{x_1} f(X_1, X_2)\right] + \sigma \mathbb{E}\left[\partial_{x_2} f(X_1, X_2)\langle DX_2, \phi_1\rangle\right].
\end{aligned}$$

Malliavin-Stein Method - Asymp. Indep.

Recall $\mathcal{N} f(x_1, x_2) = \sigma_1^2 \partial_{x_1} f(x_1, x_2) - x_1 f(x_1, x_2)$,

$$\implies \mathbb{E}\left[\mathcal{N} f(X_1, X_2)\right] = -\sigma \mathbb{E}\left[\partial_{x_2} f(X_1, X_2) \langle DX_2, \phi_1 \rangle\right],$$

and Lemma 1: $\|\partial_{x_2} f_\ell\|_\infty \leq \frac{1}{\sigma_1}\sqrt{\frac{\pi}{2}}\|\partial_{x_2}\ell\|_\infty$.

Lemma 2

If $\theta = \mathbb{P}_{X_1, X_2}$ and $\eta = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2}$ then

$$\mathrm{Wass}(\theta, \eta) \leq \frac{1}{\|\phi_1\|_2}\sqrt{\frac{\pi}{2}}\mathbb{E}\left[\,|\langle DX_2, \phi_1 \rangle|\,\right].$$

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Summing Up Part 1

- Correlation $\mathbb{E}\left[X_1 X_2\right] = \sigma \mathbb{E}\left[\langle DX_2, \phi_1 \rangle\right]$
- Distance to Independence

$$\mathrm{Wass}(\theta, \eta) \leq \frac{1}{\|\phi_1\|_2} \sqrt{\frac{\pi}{2}} \mathbb{E}\left[|\langle DX_2, \phi_1 \rangle| \right].$$

- DX_2 not only provides quantitative information about decorrelation but also about asymptotic independence.

Part 2: Proofs

KPZ Equation

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi \ , \ h_0(x) = \sigma W(x) \, .$$

Stochastic Burgers Equation

$$u_t(x) = \partial_x h_t(x) \implies \partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \xi \ .$$

Observables

$$X_1 = \int_{\mathbb{R}} \phi_1 u_0 dx = \sigma W(\phi_1) \quad \text{and} \quad X_2 = \int_{\mathbb{R}} \phi_2 u_t dx \, .$$

$\theta_t = \mathbb{P}_{X_1,X_2}$ and $\eta_t = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2}$

Recall

- $g_t(x) = \mathrm{Var}\,[h_t(x)]$;
- $\phi_2 \star \phi_1(x) = \int_{\mathbb{R}} \phi_2(z)\phi_1(z+x)dz$;
- $\psi_1' = \phi_1,\ \psi_1(0) = 0$;
- $p_t(y) = \frac{Z_t(0,y)e^{\sigma W(y)}}{\int_{\mathbb{R}} Z_t(0,z)e^{\sigma W(z)}dz}$.

Results

- $\mathbb{E}\,[p_t(y)] = \frac{g_t''(y)}{2\sigma^2}$;
- $\mathbb{E}\,[X_1X_2] = \frac{1}{2}\int_{\mathbb{R}} \phi_2 \star \phi_1(x) g_t''(x)dx$;
- $\mathrm{Wass}(\theta_t, \eta_t) \le \frac{1}{2\sigma\|\phi_1\|_2}\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}} |\phi_2'| \star |\psi_1|(x) g_t''(x)dx$.

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COMPUTING DX_2

STOCHASTIC HEAT EQUATION

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\xi \ , \ \ Z_0(x,y) = \delta(x-y)\,.$$

Cole-Hopf solution of KPZ:

$$h_t(x) = \log\left(\int_{\mathbb{R}} Z_t(x,y)e^{\sigma W(y)}dy\right)\ .$$

Computing DX_2

Thus

$$D(h_t(x))(u) = \sigma \frac{\int_{\mathbb{R}} Z_t(x,y) e^{\sigma W(y)} \mathbb{1}_y(u) dy}{\int_{\mathbb{R}} Z_t(x,y) e^{\sigma W(y)} dy},$$

where

$$u \in \mathbb{R} \mapsto \mathbb{1}_y(u) = \begin{cases} \mathbb{1}_{(0,y]}(u) & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -\mathbb{1}_{(y,0]}(u) & \text{if } y < 0. \end{cases}$$

Computing DX_2

Polymer End-Point (Y)

- Quenched $p_{x,t}(y) = \frac{Z_t(x,y)e^{\sigma W(y)}}{\int_{\mathbb{R}} Z_t(x,z)e^{\sigma W(z)}dz}$;
- $\implies D(h_t(x))(u) = \sigma E_{x,t}\left[\mathbb{1}_Y(u)\right]$;
- $X_2 = -\int_{\mathbb{R}} \phi_2'(x) h_t(x) dx$;
- $\implies DX_2(u) = -\sigma \int_{\mathbb{R}} \phi_2'(x) E_{x,t}\left[\mathbb{1}_Y(u)\right] dx$.

Computing DX_2

Polymer End-Point (Y)

- Annealed $\mathbb{E}_{x,t}[\cdot] \equiv \mathbb{E}[E_{x,t}[\cdot]]$, $\mathbb{E}_t \equiv \mathbb{E}_{0,t}$;
- Invariance $\mathbb{E}_{x,t}[f(Y)] = \mathbb{E}_t[f(Y+x)]$;
- Symmetry $\mathbb{E}_t[f(-Y)] = \mathbb{E}_t[f(Y)]$.

Computing the end-point density

Notice that

$$g_t(y) = \mathrm{Var}\left[h_t(y)\right] = \mathrm{Var}\left[h_t(0)\right] - \sigma^2|y| + 2\mathbb{E}\left[h_0(y)h_t(y)\right] .$$

IP implies that

$$\begin{aligned}
\mathbb{E}\left[h_0(y)h_t(y)\right] &= \sigma\mathbb{E}\left[\langle D(h_t(y)), \mathbb{1}_y\rangle\right] \\
&= \sigma^2\mathbb{E}\left[\langle E_{y,t}\left[\mathbb{1}_Y\right], \mathbb{1}_y\rangle\right] \\
&= \sigma^2 \begin{cases} \mathbb{E}_t\left[(y \wedge (y+Y))^+\right] & \text{if } y \geq 0\,, \\ \mathbb{E}_t\left[(y \vee (y+Y))^-\right] & \text{if } y \leq 0\,. \end{cases}
\end{aligned}$$

Computing the end-point density

$$\mathbb{E}_t\left[(y \wedge (y+Y))^+\right] = y\mathbb{P}_t\left[Y > -y\right] + \mathbb{E}_t\left[Y\mathbb{1}_{\{Y\in(-y,0]\}}\right]$$

$$\mathbb{E}_t\left[(y \vee (y+Y))^-\right] = -y\mathbb{P}_t\left[Y \le -y\right] - \mathbb{E}_t\left[Y\mathbb{1}_{\{Y\in(0,-y]\}}\right]$$

$$\left(\mathbb{E}\left[h_0(y)h_t(y)\right]\right)' = \sigma^2 \begin{cases} \mathbb{P}_t\left[Y \le y\right] & \text{if } y \ge 0\,, \\ \mathbb{P}_t\left[Y \le y\right] - 1 & \text{if } y \le 0\,, \end{cases}$$

$$\Longrightarrow \quad g_t'(y) = \sigma^2\left(2\mathbb{P}_t\left[Y \le y\right] - 1\right)\,.$$

Two-point correlation

Recall $\psi_1(y) = \int_{\mathbb{R}} \phi_1(u)\mathbb{1}_y(u)du$. Fubini implies that

$$\begin{aligned}
\langle DX_2, \phi_1\rangle &= -\sigma \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_2'(x) E_{x,t}\left[\mathbb{1}_Y(u)\right] dx \right) \phi_1(u)du \\
&= -\sigma \int_{\mathbb{R}} \phi_2'(x) E_{x,t}\left[\int_{\mathbb{R}} \phi_1(u)\mathbb{1}_Y(u)du \right] dx \\
&= -\sigma \int_{\mathbb{R}} \phi_2'(x) E_{x,t}\left[\psi_1(Y)\right] dx \,.
\end{aligned}$$

Two-point correlation

Fubini and standard integration by parts imply that

$$
\begin{aligned}
\mathbb{E}\left[\langle DX_2, \phi_1\rangle\right] &= -\sigma \int_{\mathbb{R}} \phi_2'(x)\mathbb{E}_t\left[\psi_1(x+Y)\right] dx \\
&= -\sigma \mathbb{E}_t\left[\int_{\mathbb{R}} \phi_2'(x)\psi_1(x+Y)dx\right] \\
&= \sigma \mathbb{E}_t\left[\int_{\mathbb{R}} \phi_2(x)\phi_1(x+Y)dx\right] \\
&= \sigma \mathbb{E}_t\left[\phi_2 \star \phi_1(Y)\right].
\end{aligned}
$$

Distance to independence

Similarly,

$$\begin{aligned}
\mathbb{E}\left[|\langle DX_2, \phi_1\rangle|\right] &\leq \sigma\mathbb{E}\left[\int_{\mathbb{R}} |\phi_2'(x)| E_{x,t}\left[|\psi_1(Y)|\right] dx\right] \\
&= \sigma\mathbb{E}_t\left[\int_{\mathbb{R}} |\phi_2'(x)||\psi_1(x+Y)| dx\right] \\
&= \sigma\mathbb{E}_t\left[|\phi_2'| \star |\psi_1|(Y)\right] .
\end{aligned}$$

Final Remarks

- For all $\sigma > 0$, $g_t''(x)$ gives the end-point density, the two-point correlation function, and the speed of convergence to independence Lopez-P 22;
- Analogous results can be proved with the same tools for the KPZ fixed point P 22;
- Stochastic calculus allows us to study some aspects of KPZ continuous models without the need for discrete approximations.