Probability Seminar - IM-UFRJ

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March 13, 2023

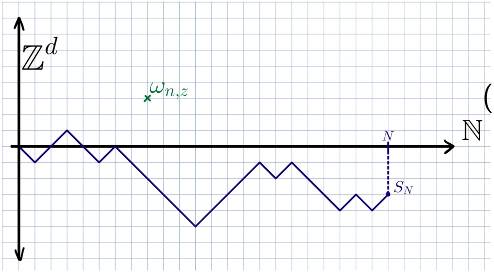
Directed Polymer in Random Environment

$$(S_0, S_1, S_2, ..., S_n, ...) : \text{ SRW on } \mathbb{Z}^d$$

$$\begin{cases} S_0 = 0, \mathrm{P}- \text{ a.s.} \\ \{S_n - S_{n-1} : n = 1, 2, 3, \dots\} \text{ i.i.d. sequence} \\ \mathrm{P}[S_1 = e_j] = \mathrm{P}[S_1 = -e_j] = 1/2d, \forall j, \{e_1, \dots, e_d\} \text{canonical basis} \end{cases}$$

$$\{\omega_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\} : \text{i.i.d. random variables}$$

$$\text{with } \mathbb{P}[e^{\beta\omega_{n,z}}] < \infty, \forall \beta \in \mathbb{R}$$



$$(S_0, S_1, ..., S_N) \mapsto e^{\beta \sum_{n=1}^N \omega_{n,S_n}}$$

$$\frac{d\mathrm{P}^\omega_{N,\beta}}{d\mathrm{P}} = \frac{e^{\beta \sum_{n=1}^N \omega_{n,S_n}}}{Z_N^{\beta,\omega}}$$

$$Z_N^{\beta,\omega} = \mathrm{E}[e^{\beta \sum_{n=1}^N \omega_{n,S_n}}].$$

Overview of known results

- The sequence

$$W_N^{\beta,\omega} := \frac{Z_N^{\beta,\omega}}{\mathbb{E}\left[Z_N^{\beta,\omega}\right]}, \tag{1}$$

  is a positive martingale with respect to the filtration $\mathcal{G}_N := \sigma\{\omega_{n,z} : 1 \leq n \leq N, z \in \mathbb{Z}\}$.
- The limit

$$W_\infty^{\beta,\omega} := \lim_{N\to\infty} W_N^{\beta,\omega}, \tag{2}$$

  exists $\mathbb{P}$-*a.s.* and is a non-negative random variable.
- The event $\{W_\infty^{\beta,\omega} = 0\}$ belongs to the tail σ-field $\mathcal{G}_N$

$$\text{Weak Disorder} \iff W_\infty^\beta > 0 \ \mathbb{P}\text{-}a.s.$$
$$\text{Strong disorder} \iff W_\infty^\beta = 0 \ \mathbb{P}\text{-}a.s.$$

Overview of known results

- (Comets, Yoshida 2006) There exists a critical value $\beta_c = \beta_c(d) \in [0, \infty]$ such that:

$$\begin{aligned} \text{Weak Disorder} &\iff \beta \in [0, \beta_c) \\ \text{Strong disorder} &\iff \beta \in (\beta_c, \infty) \end{aligned}$$

Free energy

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\omega} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_N^{\beta,\omega}. \qquad (3)$$

Known facts about the free energy:

- $F(\beta) - \lambda(\beta) < 0 \implies \lim_{N\to\infty} W_N^{\beta,\omega} = 0$

  $$\text{Very strong disorder} \iff F(\beta) - \lambda(\beta) < 0$$

- (Comets, Shiga, Yoshida 2003) $p(\beta) := F(\beta) - \lambda(\beta)$ is continuous and non-increasing. Thus, there is a critical value $\bar{\beta}_c$ such that

  $$p(\beta) = 0 \iff \beta \in [0, \bar{\beta}_c]$$
  $$p(\beta) < 0 \iff \beta \in (\bar{\beta}_c, \infty)$$

- $\beta_c \leq \bar{\beta}_c$

Weak disorder regime

- If $\lambda(2\beta) - 2\lambda(\beta) < \log \frac{1}{\pi_{\mathrm{P}}}$, weak disorder holds, where

$$\pi_{\mathrm{P}} := \mathrm{P} \otimes \mathrm{P}' \left[\exists n \geq 1 : S_n = S'_n \right]$$

- (Comets, Yoshida 2006) Assume P is the SRW, $d \geq 3$ and weak disorder. Then, for all bounded continuous functions f on the path space, we have the following convergence in probability

$$\mathrm{E}_N^{\beta,\omega} \left[f(\widehat{S}^{(N)}) \right] \xrightarrow{\mathbb{P}} \mathrm{E}\left[f(W) \right],$$

as $N \to \infty$ where $\widehat{S}^{(N)} := \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}}, t \in [0,1]$ and W is the Brownian motion with diffusion matrix $d^{-1} I_d$.

Strong and very strong disorder regime

- For any $d \geq 1$, we have $p(\beta) < 0$ whenever

$$\beta\lambda'(\beta) - \lambda(\beta) > -\sum_{x\in\mathbb{Z}^d} \mathrm{P}\left[S_1 = x\right] \log \mathrm{P}\left[S_1 = x\right]. \qquad (4)$$

- For $\beta > 0$,

$$\left\{W_\infty^{\beta,\omega} = 0\right\} = \left\{\sum_{n\geq 1} \left(\mathrm{P}_{n-1}^{\beta,\omega}\right)^{\otimes 2} \left[S_n = S'_n\right] = \infty\right\} \ \mathbb{P}\text{-a.s.}, \qquad (5)$$

where S and S' are two independent polymers with distribution $\mathrm{P}_{n-1}^{\beta,\omega}$. Moreover, if $\mathbb{P}[W_\infty^{\beta,\omega} = 0] = 1$, then there exists some constants c_1, $c_2 \in (0,\infty)$ such that,

$$-c_1 \log W_N^{\beta,\omega} \leq \sum_{n\geq 1}^{N} \left(\mathrm{P}_{n-1}^{\beta,\omega}\right)^{\otimes 2} \left[S_n = S'_n\right] \leq -c_2 \log W_N^{\beta,\omega}, \qquad (6)$$

for N large enough, $\mathbb{P}$-a.s.

Free energy asymptotics at high temperature.

- (Nakashima 2019) In $d = 1$, $p(\beta)$ is of order $-\beta^4$ as $\beta \to 0$ and, under some conditions on the environment,

$$\lim_{\beta \to 0} \frac{p(\beta)}{\beta^4} = -\frac{1}{6}. \tag{7}$$

- (Berger, Lacoin 2017) In $d = 2$, $p(\beta)$ is smaller than any power of β at the neighborhood of 0, specifically

$$\lim_{\beta \to 0} \beta^2 \log |p(\beta)| = -\pi. \tag{8}$$

Directed polymer with very heavy tailed random walks

$$\mathrm{P}\left[X_1 = n\right] =: K(n) = \frac{L(n)}{n}, \tag{9}$$

where $L(\cdot)$ is a *slowly varying function* at ∞. That is, for all $k > 0$

$$\lim_{n\to\infty} \frac{L(kn)}{L(n)} = 1. \tag{10}$$

Theorem (2013)

Assuming that the distribution of the increments satisfies (9) *and that* $K(n) > 0$ *for all* $n \in \mathbb{Z}$ *then,*

$$p(\beta) = 0, \tag{11}$$

for all $\beta \in \mathbb{R}$, *which implies that there is no very strong disorder regime.*

Theorem (2013)

If the distributions of the increments and the environment satisfy

$$\beta\lambda'(\beta) - \lambda(\beta) > -\sum_{n\geq 1} K(n)\log K(n), \tag{12}$$

for some $\beta > 0$ *then* $\beta_c < \beta$. *In particular, there is a strong disorder phase whenever*

$$\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = \infty \quad \textit{and} \quad \sum_{n\geq 1} K(n)\log(1/K(n)) < \infty.$$

Notice that if

$$K(n) = \frac{(\log\log|n|)^{\alpha}}{|n|(\log|n|)^2}(1+o(1)) \tag{13}$$

where $\alpha < -1$, the polymer presents a strong disorder phase with $p(\beta) = 0$

Theorem (2013)

Under the following conditions on the law of the increments:

(a) $K(\cdot)$ *is unimodal and symmetric around 0 (that is,* $K(n) = K(-n)$ *and* $K(n) \geq K(n+1)$ *for all* $n \geq 0$*)*

(b)

$$K(n) \geq \frac{(\log \log n)^{\alpha}}{n(\log n)^2}, \tag{14}$$

for all n sufficiently large, and some $\alpha > 1$*,*

(c) *and*

$$\frac{\mathrm{P}\left[X_1 \in (s_n, 2ns_n)\right]}{\mathrm{P}\left[X_1 \geq s_n\right]} \leq \frac{1}{n^{\gamma}}, \tag{15}$$

where $\gamma > \frac{1}{2}$ *and*

$$s_n := \min\left\{ s \in \mathbb{N} : \mathrm{P}\left[X_1 \geq s\right] \leq \frac{(\log n)^2}{n} \right\}, \tag{16}$$

for all n sufficiently large, then, $\beta_c = \infty$*.*

Assuming that the environment is unbounded form above, we conjecture that the following equivalence

$$\beta_c < \infty \quad \Leftrightarrow \quad \sum_{n\geq 1} K(n) \log \frac{1}{K(n)} < \infty. \tag{17}$$

is true.

Other problems:

- Random walks over particle systems.