Quantitative results in Poincaré recurrence for mixing processes

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Thanks to Giulio and Eulália for the invitation!

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Based on recent works with M. Abadi (IME-USP), V. Amorim (IFSP), JR. Chazottes (Polytechnique), N. Haydn (USC), S. Vaienti (Aix-Marseille).

Présentation

Dynamical system notation is easier for this talk:

Consider (Ω, T, μ), $T : \Omega \to \Omega$, and a T-invariant probability measure $\mu \circ T^{-1} = \mu$.

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To fix idea: think about an i.i.d. sequence of Ber(p) r.v.'s!!

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*T-**invariance** $\Rightarrow \mu_B(\tau_B(x) < \infty) = 1$ for B with $\mu(B) > 0$.*
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What about $\mu_B(\tau_B \mu(B) > t)$?

Theorem (Folklore)

For suitable dynamics and suitable $B_n(x) \to x \in \Omega$

$$\mu_{B_n}(\tau_{B_n}\mu(B_n) > t) \stackrel{n\to\infty}{\longrightarrow} e^{-t} \quad , \quad \underline{\mu - a.e.\ x \in \Omega.}$$

The scaling factor $\mu(B_n)$ comes from the Kac Lemma.

"suitable dynamics": mixing conditions (stronger than ergodicity)

"suitable $B_n(x)$": Think about "cylinders/patterns".

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For suitable dynamics and suitable $B_n(x) \to x \in \Omega$

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The new scaling parameter $\lambda(B_n(x))$ accounts for periodic x's.

And $\lambda(B_n(x)) \to 1$ for aperiodic x's.

Abadi *et al.* (2000's): Several possible choices for $\lambda(B_n(x))$, but a simple one is

$$\mu_{B_n}(\tau_{B_n} > \text{“first possible return to } B_n\text{”})$$

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For suitable dynamics and cylinder sets or balls $B_n(x) \to x \in \Omega$

$$\mathcal{L}(N_n) \Rightarrow \text{Pólya-Aeppli}(t\lambda) \quad , \quad \forall x \in \Omega$$

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Pólya-Aeppli($t\lambda$) is the Compound Poisson $\sum_{i=1}^{M} X_i$ with

- $X_i' s$ are i.i.d. Geo(λ)
- $M \sim$ Pois($t\lambda$).

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- G., Haydn and Vaienti (to appear, 2023)
  -> for symbolic dynamical systems.

  Motivation: approximate synchronization of stochastic processes.

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For ergodic measure μ

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What about the fluctuations?

Theorem (Collet, Galves and Schmitt (1999))

If μ satisfies

- *strongly mixing,*
- *with $\sigma^2 := \lim Var\left(\frac{\log r_n}{\sqrt{n}}\right) > 0$,*

then:

$$\mathcal{L}\left(\frac{\log r_n/n - h_\mu}{\sigma/\sqrt{n}}\right) \Rightarrow \mathcal{N}(0,1),$$

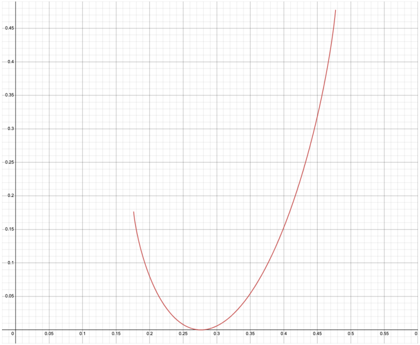
and there exists a $u_0 > 0$ s.t. for $u \in [0, u_0[$

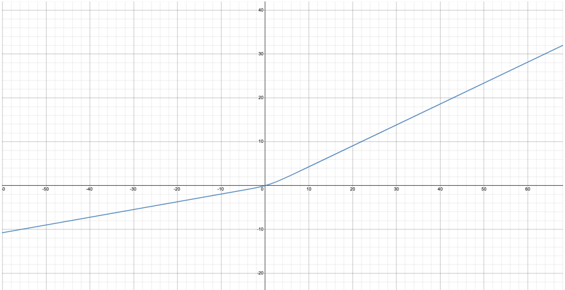
$$-\lim \frac{1}{n} \log \mu\left(\frac{\log r_n}{n} > h + u\right) = m(h+u)$$
$$-\lim \frac{1}{n} \log \mu\left(\frac{\log r_n}{n} < h - u\right) = m(h-u),$$

where m is the Legendre-Fenchel transform of

$$\mathcal{M}(q) := \lim_n \frac{1}{n} \log \mathbb{E}\left(\mu(X_1^n)^{-1}\right)^q \ , \quad q \in \mathbb{R}.$$

Observation: Example for i.i.d. Ber(1/3)





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Theorem (Abadi, Amorim, Chazottes, G. (to appear, 2023))

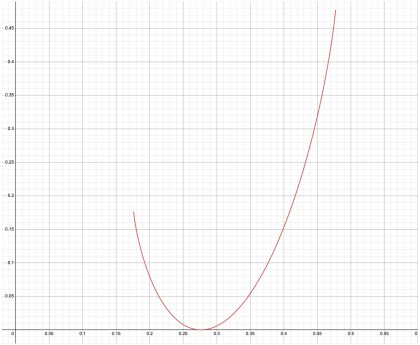
Same conditions as before:

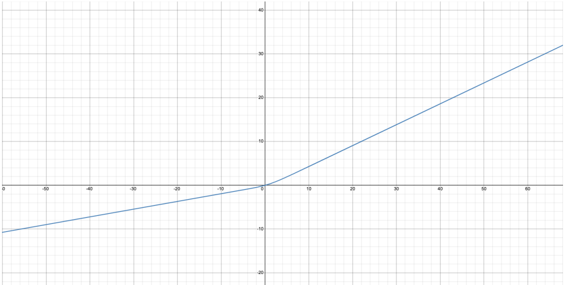
- Λ *exists and equals* $\lim_n \frac{1}{n} \log \max_{a_0^{n-1}} \mu(a_0^{n-1})$
- *and* $\mathcal{R}(q)$ *exists and*

$$\begin{aligned} \mathcal{R}(q) &= \mathcal{M}(q) \quad \textit{for } q > q^\star \\ &= \Lambda \qquad\quad \textit{for } q \leq q^\star \end{aligned}$$

where $q^\star \in (-1, 0)$ *is the solution of the equation* $\mathcal{M}(q) = \Lambda$*.*

Consequence:





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These **local** considerations yield “ansatz”

$$r_n \longleftrightarrow \mu(X_1^n)^{-1}$$

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Haydn *et al.* (2002) shown, by simulations, that this doesn't work.

THANK YOU!!!