Oriented Percolation Models: Further improvement for the critical probability on regular trees

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Outline

An oriented percolation model on regular trees

- $\mathbb{T}_d = (V, E)$ - regular tree of degree $d + 1 \geq 3$ with root o.
- $\mathbb{K}_d = (V, \vec{E}_d)$ - complete, oriented graph which vertex set equals V.

$$\vec{E}_d = \{(x, y) : x, y \in V\}$$

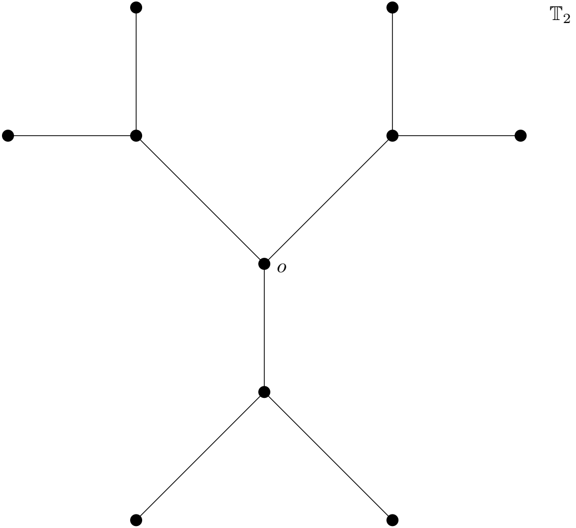
- For $x, y \in V$,

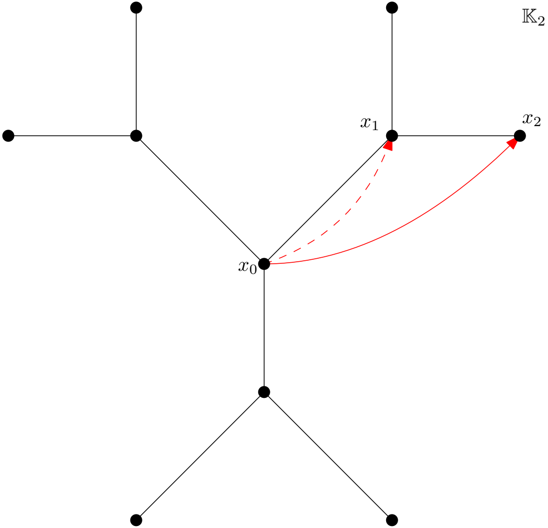
$$\{x \to y\} = \{\text{oriented bond from } x \text{ to } y \text{ is open}\}$$

- For $C_d = C > 0$ (constant) e $\rho = \rho \in [0, 1]$ (fixed parameter), define

$$\mathbb{P}(x \to y) = C\ \rho^{\text{dist}(x,y)} \tag{1}$$

- For $x, y, z \in V$, we assume $\{x \to y\} \subset \{x \to z\}$, if y belongs to the path connecting x and z.


$\mathbb{T}_2$
o


$\mathbb{K}_2$
x_1
x_2
x_0

A toy model for infectious disease

Disk percolation model

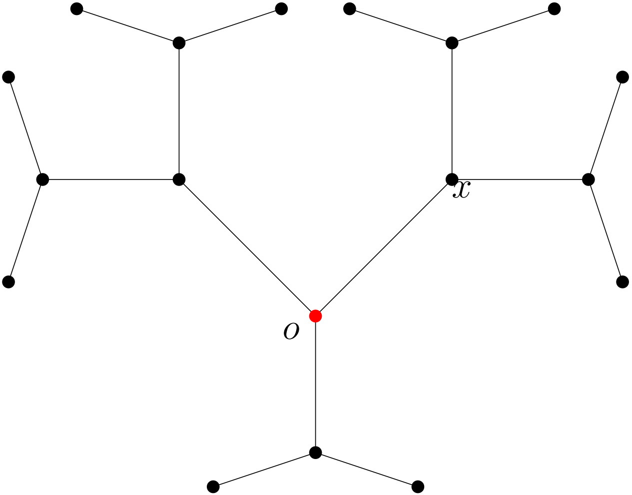
1. Each vertex $x \in V$ has associated an *infection radii* $R_x \in \{0, 1, \ldots\}$, such that $R_x \sim Geom(1-p)$.
2. There are infected and healthy vertices.
3. At time zero, only the root is infected.
4. Each infected vertex remain infected forever.
5. For $x, y \in V$, $\{x \to y\} \iff \text{dist}(x,y) \le R_x$. Then:

$$\mathbb{P}(x \to y) = p^{\text{dist}(x,y)}$$

6. Hence, we have $C = 1$ e $\rho = p$.

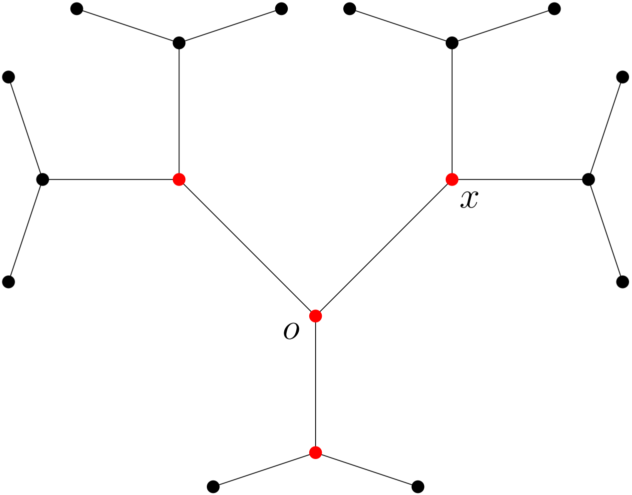
$n = 0$

$R_o = 1$

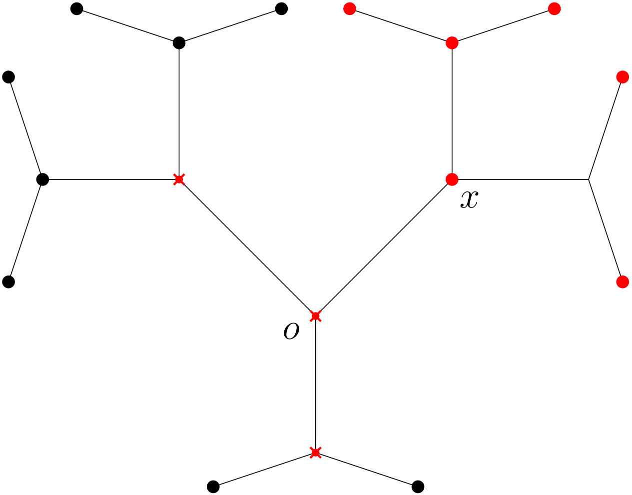


$n = 1$

$R_x = 2$

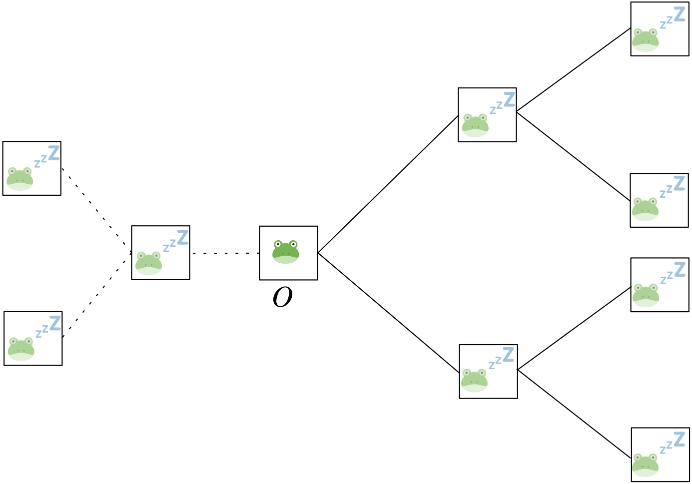


$n = 2$



Frog model

1. There are sleeping and awake *frogs* at the vertices of the tree.
2. At time zero, each vertex is occupied by a *frog* and only the frog at the root is awake.
3. Awake frogs move as a discrete-time simple random walks on the tree and disappears after a random number of steps, $\tau \in \{0, 1, \ldots\}$, such that $\tau \sim Geom(1-p)$.
4. Each time that a vertex containing a sleeping frog is visited by an awake frog; the sleeping frog is woken up and start its own life and trajectory independently of anything else.
5. $\{x \to y\}$ if y belongs in the *range* of the frog initially located at x.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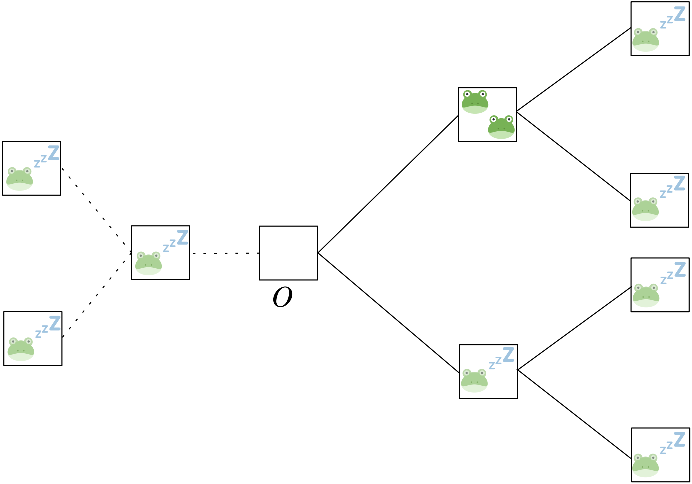
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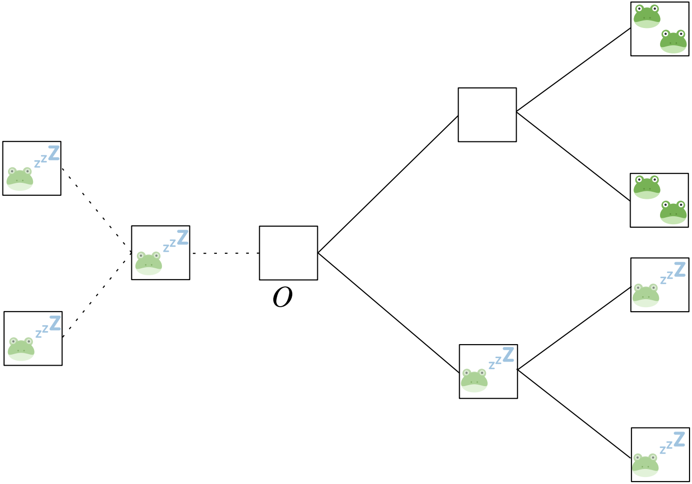
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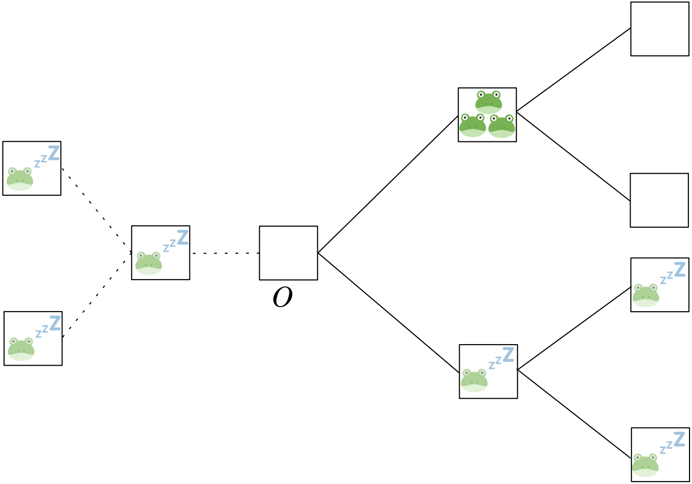
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Self-avoiding frog model

- *Same rules of the (usual) frog model, with the exception that each simple random walk is replaced by a self-avoiding random walk.*

So, for the frog model, we have:

$$\mathbb{P}(x \to y) = \left(\frac{d + 1 - \sqrt{(d+1)^2 - 4d\,p^2}}{2d\,p} \right)^{\mathrm{dist}(x,y)}, \qquad (2)$$

and, for the self-avoiding frog model we have:

$$\mathbb{P}(x \to y) = \left(\frac{d}{d+1} \right) \left(\frac{p}{d} \right)^{\mathrm{dist}(x,y)}. \qquad (3)$$

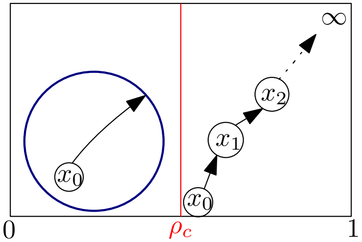
Thus, $C = 1$, $\rho = \left(\frac{d+1-\sqrt{(d+1)^2-4d\,p^2}}{2d\,p} \right)$ [for the FM],

and $C = \left(\frac{d}{d+1} \right)$, $\rho = \left(\frac{p}{d} \right)$ [for the SAFM].

Critical probability

The main quantity of interest here, is the *critical probability*:

$$\rho_c(\mathbb{T}_d) = \inf\{\rho : \mathbb{P}(o \to \infty) > 0\}.$$



Previous upper bound for the critical probability

Theorem (Lebensztayn & Utria (2015))

For any $d \geq 2$, we have

$$\rho_c(\mathbb{T}_d) \leq \bar{\rho}_{\mathbf{old}}(d),$$

where $\bar{\rho}_{\mathbf{old}}(d)$ is the unique root on [0,1/d] of the polynomial

$$Q^{(d)}(\rho) = dC\rho^2 - d(1+C)\rho + 1.$$

Further improvement for the upper bound

Theorem (Main Result)

For any $d \geq 2$, we have
(i) $\rho_c(\mathbb{T}_d) \leq \bar{\rho}_{\mathbf{new}}(d)$, *where* $\bar{\rho}_{\mathbf{new}}(d)$ *is the unique root in [0,1/d] of the polynomial*

$$P^{(d)}(\rho) = 2C((1-C)+d^2(C+1))\rho^4 - 4Cd(Cd+1)\rho^3 + 4d(C+1)\rho - 4.$$

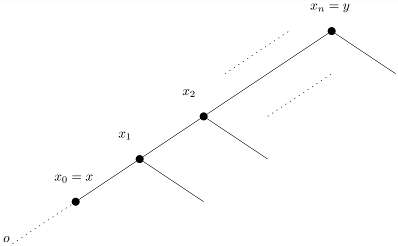
(ii) $\bar{\rho}_{\mathbf{new}}(d) < \bar{\rho}_{\mathbf{old}}(d)$.

Strategy of the proof

1. For each $n \geq 1$. Define a branching processes *below* the oriented percolation model, in the sense, that if the former survives, the latter percolates.
2. Obtain a sufficient condition (as a function of ρ and d) for the survival of the branching process, this leaves to a sequence of upper bounds for the critical probability.
3. Prove that the sequence of upper bounds in the previous step converges to the upper bound $\bar{\rho}_{\mathbf{new}}(d)$ as $n \to \infty$.

Definition

- For $x \in V$, define $L_n(x)$ as the n-th *level* of the tree starting from x.
- For $x \in V$ and $y \in L_n(x)$, let $x = x_0, x_1, \ldots, x_n = y$ be the vertices in the path connecting x and y, in such a way that $x_\ell \sim x_{\ell+1}$ For each $\ell = 1, 2, \ldots, n-1$, let $[x_0 \rightsquigarrow x_\ell]$ denote the event that $[x_0 \to x_\ell] \cap [x_0 \nrightarrow x_{\ell+1}]$.



Now we define the event $[x_0 \stackrel{mother}{\rightarrow} x_n]$ inductively on n by:

(i) If $n = 1$, then $[x_0 \stackrel{mother}{\rightarrow} x_1] = [x_0 \rightarrow x_1]$.

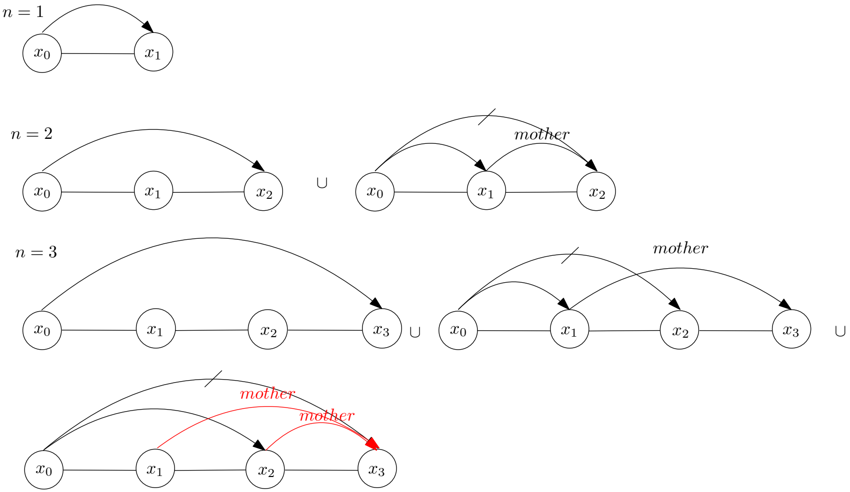
(ii) If $n = 2$, then
$[x_0 \stackrel{mother}{\rightarrow} x_2] = [x_0 \rightarrow x_2] \cup [x_0 \rightsquigarrow x_1, x_1 \stackrel{mother}{\rightarrow} x_2]$.

(iii) If $n \geq 3$, then

$$[x_0 \stackrel{mother}{\rightarrow} x_n] = [x_0 \rightarrow x_n] \cup [x_0 \rightsquigarrow x_1, x_1 \stackrel{mother}{\rightarrow} x_n] \cup \bigcup_{\ell=2}^{n-1} [x_0 \rightsquigarrow x_\ell, \langle x_{\ell-1}, x_\ell \rangle \stackrel{mother}{\rightarrow} x_n],$$

where

$$[\langle x_{\ell-1}, x_\ell \rangle \stackrel{mother}{\rightarrow} x_n] = [x_{\ell-1} \stackrel{mother}{\rightarrow} x_n] \cup [x_\ell \stackrel{mother}{\rightarrow} x_n].$$


$n = 1$
x_0
x_1
$n = 2$
x_0
x_1
x_2
$\cup$
mother
$n = 3$
x_3
$\cup$
mother
$\cup$
mother
mother

Branching Processes below the oriented percolation model

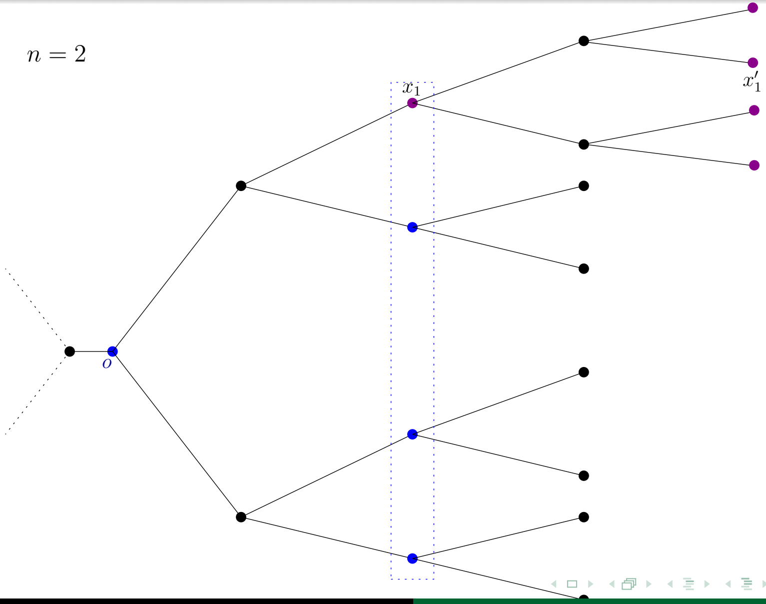
Definition

For $n \geq 1$. Consider

$$\xi_n = \sum_{x_n \in L_n(x_0)} \mathbb{1}\{x_0 \overset{mother}{\to} x_n\}.$$

Define a branching process starting with the root o and whose progeny is given by the random variable ξ_n.

$n = 2$



Computing the progeny probability

Let $\phi_n(\rho) = \mathbb{P}(x_0 \overset{mother}{\rightarrow} x_n)$ be the probability of x_0 be mother of x_n, then we have:

$$
\begin{aligned}
\phi_1(\rho) &= C\rho, \\
\phi_2(\rho) &= C\rho^2\left(1 + C(1-\rho)\right), \\
\phi_n(\rho) &= C\rho^n + C\rho(1-\rho)\phi_{n-1}(\rho) \\
&\quad + \sum_{j=2}^{n-1} C\rho^j(1-\rho)\left[\phi_{n-j}(\rho) + (1-C\rho)\phi_{n-j}(p)\right], n \geq 3.
\end{aligned}
$$

Auxiliary results

Lemma

The probability $\phi_n(\rho)$, satisfies the following linear difference equation of second order:

$$\phi_n(\rho) = \rho(1 + C(1-\rho^2))\phi_{n-1}(\rho) - C^2\rho^3(1-\rho)\phi_{n-2}(\rho), n \geq 3,$$

w.i.c., $\phi_1(\rho) = C\rho$ *and* $\phi_2(\rho) = C\rho^2(1 + C(1-\rho))$.

Lemma

For any $0 < \rho \leq 1/d$, *we have*

$$\lim_{n\to\infty} [\phi_n(\rho)]^{1/n} = \lambda(\rho) = \frac{\rho}{2}[1 + C(1-\rho^2) + \psi(\rho)], \tag{4}$$

where $\psi(\rho) = \sqrt{C^2\rho^4 - 2C(1-C)\rho^2 - 4C^2\rho + (C+1)^2}$.

- Since the mean number of children per individual in the branching process just defined is $\mu_n(\rho) = d^n \phi_n(\rho)$, then we have to find a value $\rho_n(d)$ of ρ sufficiently large such that for $\rho > \rho_n$, $\mu_n(\rho) > 1$.
- Equivalently, $\Gamma_n^{(d)}(\rho) = [\phi_n(\rho)]^{1/n} - 1/d > 0$

Lemma (Properties of $\Gamma_n^{(d)}(\rho)$)

- $\Gamma_n^{(d)}(0) < 0$ *and* $\Gamma_n^{(d)}(1) > 0$
- $\Gamma_n^{(d)}(\rho)$ *is increasing in* ρ.

Consequently, there exists a unique root, $\rho_n(d)$, in [0,1/d] such that for $\rho > \rho_n(d)$, we have $\mu_n(d) > 1$.

So, we have that

$$\rho_c(\mathbb{T}_d) \leq \rho_n(d). \tag{5}$$

Finally, by the preceding auxiliary results, we conclude that

$$\lim_{n\to\infty} \Gamma_n^{(d)}(\rho) = \frac{\rho}{2}[1 + C(1-\rho^2) + \psi(\rho)] - 1/d = \Gamma^{(d)}(\rho) \tag{6}$$

The result follows from the following fact:

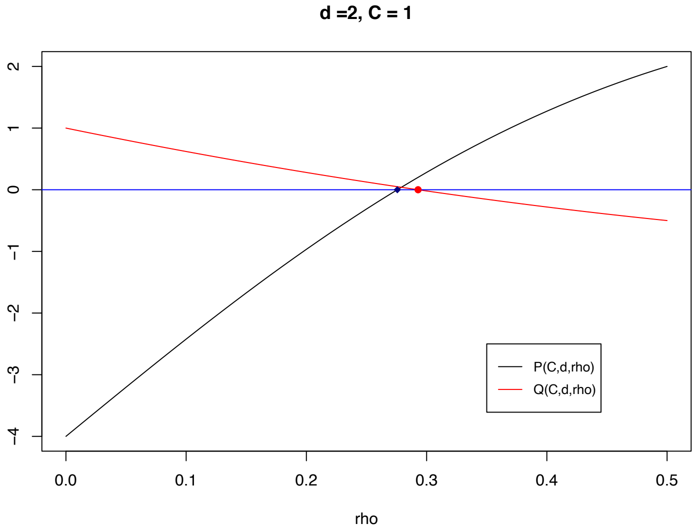
Proposition

Assume (f_n) is a sequence of increasing continuous functions and that $f_n \to f$, such that f is also an increasing and continuous function. If $f_n \to f$ and $f_n(\theta_n) = 0$, then $\theta_n \to \theta$ in which θ satisfies $f(\theta) = 0$.

- Noting that,

$$\Gamma^{(d)}(\rho) = 0 \iff P^{(d)}(\rho) = 0, \tag{7}$$

  we obtain the first part of our main result.
- For the second part, it is enough to check that $P^{(d)}(\bar{\rho}_{\mathbf{old}}(d)) > 0$ for all $C > 0$ and $d \geq 2$.


d =2, C = 1
2
1
0
−1
−2
−3
−4
0.0
0.1
0.2
0.3
0.4
0.5
rho
P(C,d,rho)
Q(C,d,rho)

Further improvement for the DPM

Theorem (Lebensztayn & Rodriguez (2008))

For any $d \geq 2$ the critical probability of the disk percolation model satisfies

$$p_c^{\mathbf{dpm}}(\mathbb{T}_d) \leq \hat{p}(d) = 1 - \sqrt{1 - 1/d}. \tag{8}$$

Corollary

For any $d \geq 2$, we have

(i) $p_c^{\mathbf{dpm}}(\mathbb{T}_d) \leq \bar{p}(d)$,

(ii) $\bar{p}(d)$ *is the unique root of the polynomial*

$$Q_1^{(d)}(p) = d^2p^4 - d(d+1)p^3 + 2dp - 1.$$

(iii) $\bar{p}(d) < \hat{p}(d)$.

Consequences concerning to the FM

Corollary (Lebensztayn & Utria(2019))

(i) $p_c^{\mathbf{fm}}(\mathbb{T}_d) \leq \bar{p}(d)$.

(ii) $\bar{p}(d)$ *is the unique root in (0,1) of the polynomial*

$$Q_2^{(d)}(p) = p^4 - \frac{4(d+1)}{(3d+1)}p^3 - \frac{2(d-1)(d+1)^2}{(3d+1)^2}p^2 + \frac{(d+1)^3}{d(3d+1)}p - \frac{(d+1)^4}{d(3d+1)^2}.$$

Remarks.

(a) $\bar{p}(d)$ was an improvement of previous bounds obtained in Lebensztayn, Machado & Popov (2005); Gallo & Rodríguez (2018).

Consequences concerning to the FM

Corollary (Lebensztayn & Utria(2019))

(i) $p_c^{\mathbf{fm}}(\mathbb{T}_d) \leq \bar{p}(d)$.

(ii) $\bar{p}(d)$ *is the unique root in (0,1) of the polynomial*

$$Q_2^{(d)}(p) = p^4 - \frac{4(d+1)}{(3d+1)}p^3 - \frac{2(d-1)(d+1)^2}{(3d+1)^2}p^2 + \frac{(d+1)^3}{d(3d+1)}p - \frac{(d+1)^4}{d(3d+1)^2}.$$

Remarks.

(a) $\bar{p}(d)$ was an improvement of previous bounds obtained in Lebensztayn, Machado & Popov (2005); Gallo & Rodríguez (2018).

(b) It is worth noting that $\bar{p}(d)$ was recently improved in Gallo & Pena (2022).

Further improvement for the SAFM

Theorem (Gallo & Rodríguez(2018))

For any $d \geq 3$, the critical probability of the SAFM satisfies:

$$p_c^{\mathbf{safm}}(\mathbb{T}_d) \leq \tilde{p}(d) = (d+1)^2 \frac{F(c,d) - \sqrt{F^2(c,d) - 224c^2(c+1)^2}}{16d}, \tag{9}$$

where $F(c,d) = 7d(c+1)^3 - 8c^2$.

Corollary

(i) *If* $d \geq 2$, *then* $p_c^{\mathbf{safm}}(\mathbb{T}_d) \leq \bar{p}(d)$,

(ii) *is the unique root in (0,1) of the polynomial*

$$Q_3^{(d)}(p) = \frac{2(d^2-1)(2d+1)}{d^3(d+1)^2}p^4 - \frac{4(d^2+d+1)}{d(d+1)^2}p^3 + \frac{4(2d+1)}{d+1}p - 4$$

(iii) *If* $d \geq 4$, *then* $\bar{p}(d) < \tilde{p}(d)$.

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