A lower bound for set colouring Ramsey numbers

Taísa Martins (UFF)

with L. Aragão, M. Collares, J. P. Marciano and R. Morris

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Ramsey number

\[ R_r(K) = \min \{ n \mid \forall c : E(K_n) \to [r] \text{ has a monochromatic } K_k \} \]

ex.: \[ R_2(3) \]

\[ R_2(3) > 5 \]
Ramsey number

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ex.: \( R_2(3) \)

\[ R_2(3) > 5 \]
Ramsey number

\[ R_r(K) = \min \{ n \mid \forall c : E(K_n) \to [r] \text{ has a monochromatic } K_k \} \]

ex.: \( R_2(3) = 6 \)

\[ R_2(3) > 5 \]

\[ R_2(3) \leq 6 \]
Set colouring Ramsey

Set colouring Ramsey number

\[ R_{r,s}(K) = \min \{ n \mid \forall c : E(K_n) \rightarrow ([r]) \text{ has a monochromatic } K_k \} \]
Set colouring Ramsey

Set colouring Ramsey number

\[ R_{r,s}(k) = \min \{ n \mid \forall c : E(K_n) \rightarrow ([r]_s) \text{ has a monochromatic } K_k \} \]

\[ \Rightarrow \]

\[ \exists \text{ a common colour to all edges:} \]

\[ \{1,2\} \quad \{1,2\} \quad \{1,3\} \]
Set colouring Ramsey

Set colouring Ramsey number

\[ R_{r,s}(k) = \min \{ n \mid \forall \, c : E(K_n) \to [r]^s \text{ has a monochromatic } K_k \} \]

\( s = 1 \):

\[ R_{r,1}(k) = R_r(k) \]

best known bounds

\[ 2^{\log_2(kr)} \leq R_r(k) \leq r^{O(kr)} \]
Ramsey bound

upper bound (Erdős - Szekeres) \( R_{2,3}(k) \leq 4^k \)
Ramsey bound

upper bound (Erdős - Szekeres) \( R_{2,1}(k) \leq 4^k \)

Let \( c \) be a 2-colouring of \( E(K_n) \) where \( n = 4^k \) has a mono X clique.

proof:

Let \( c \) be a 2-colouring
with no \( K_k \) mono X
Ramsey bound

Upper bound (Erdős - Szekeres) \( R_{2,3}(k) \leq 4^k \)

Let \( X \) be a 2-colouring of \( E(K_n) \) where \( n = 4^k \) has a mono \( X \) clique.

Proof:

Let \( c \) be a 2-colouring with no \( K_k \) mono \( X \):

\[ X_0 = V \]
Ramsey bound

upper bound (Erdős - Szekeres) \( R_{2,3}(k) \leq 4^k \)

Let \( \text{wts} \): Any 2-colouring of \( E(K_n) \) where \( n = 4^k \) has a monochromatic clique.

proof:

Let \( c \) be a 2-colouring with no \( K_k \) monochromatic:

\[
V \ni x \quad \Rightarrow \quad \left\lceil \frac{n-1}{2} \right\rceil
\]

\( X_0 = V \)
Ramsey bound

upper bound (Erdős - Szekeres) \( R_{2,1}(K) \leq 4^k \)

Let WTS: Any 2-colouring of \( E(K_n) \) where \( n = 4^k \) has a mono \( K_k \) clique.

proof:

Let \( c \) be a 2-colouring with no \( K_k \) mono \( K \):

\[
\begin{align*}
X_0 &= V \\
X_1 &= X_0 \cap N_{B}(v_j)
\end{align*}
\]
Ramsey bound

Upper bound (Erdős - Szekeres) \( R_{2,3}(n) \leq 4^k \)

Let \( \text{WTS: Any 2-colouring of } E(K_n) \text{ where } n = 4^k \text{ has a monochromatic clique.} \)

Proof:

Let \( c \) be a 2-colouring with no \( k_k \) monochromatic:

\[ X_0 = V \]

\[ X_1 = X_0 \cap N_B(v_j) \]
Ramsey bound

upper bound (Erdős - Szekeres) \( R_{2,1}(k) \leq 4^k = 2^{2k} \)

\[ x_i = x_{i-1} \cap \bigcap_{v \in B} B(v) \cap \bigcap_{v \in R} R(v) \]
Ramsey bound

Upper bound (Erdős - Szekeres)

\[ R_{2,1}(k) \leq 4^k = 2^{2^k} \]

\[ |X_0| = |V| = n \]

\[ |X_1| \geq (n-1)/2 = \frac{n}{2} - \frac{1}{2} \]

\[ |X_2| \geq (|X_2| - 1)/2 = \frac{n}{4} - \frac{1}{2} - \frac{1}{4} \]

\[ \vdots \]

\[ |X_i| \geq \frac{n}{2^i} - 1 \]

\[ X_i = X_{i-1} \cap \bigcap_{v \in B} N_B(v) \cap \bigcap_{v \in R} N_R(v) \]
Ramsey bound

upper bound (Erdős - Szekeres)

\[ R_{2,1}(k) \leq 4^k = 2^{2k} \]

\[ |X_0| = |V| = n \]
\[ |X_1| \geq (n-1)/2 = \frac{n}{2} - \frac{1}{2} \]
\[ |X_2| \geq (|X_1|-1)/2 \geq \frac{n}{4} - \frac{1}{2} - \frac{3}{4} \]
\[ \vdots \]
\[ |X_i| \geq \frac{n}{2^i} - 1 \]

When \( i = 2(k-1) \),
\[ |X_i| \geq \frac{4^k}{2^{2(k-1)}} - 1 \]
Ramsey bound

**Upper bound (Erdős - Szekeres)**

\[ R_{2,1}(k) \leq 4^k = 2^{2k} \]

- \[ |X_0| = |V| = n \]
- \[ |X_1| > (n-1)/2 = \frac{n}{2} - \frac{1}{2} \]
- \[ |X_2| > (|X_2| - 1)/2 = \frac{n}{4} - \frac{1}{2} - \frac{1}{4} \]
- \[ \vdots \]
- \[ |X_i| > \frac{n}{2^i} - 1 \]

When \( i = 2(k-1) \),

\[ |X_i| > \frac{4^k}{2^{2(k-1)}} - 1 > 1 \]

we have a monox \( K_k \!\!\!.\!\!

Ramsey bound

upper bound (Erdős - Szekeres)

\[ R_{2,1}(k) \leq 4^k = 2^{2k} \]

\[ |X_0| = |V| = n \]

\[ |X_1| \geq \frac{(n-1)/2}{2} = \frac{n}{2} - \frac{1}{2} \]

\[ |X_2| \geq \frac{|x_1||V_2|}{2} \geq \frac{n}{4} - \frac{1}{2} - \frac{3}{4} \]

\[ \vdots \]

\[ |X_i| \geq \frac{n}{2^i} - 1 \]

When \( i = 2(k-1) \),

\[ |X_i| \geq \frac{4^k}{2^{2(k-1)}} - 1 \geq 1 \]

we have a monox \( K_k \)!
Ramsey bound

**Upper Bound (Erdős - Szekeres)**

\[ R_{r,3}(k) \leq r^k \]

\[ \left| X_0 \right| = \left| V \right| = n \]
\[ \left| X_1 \right| = \frac{(n-1)/2}{2} = \frac{n}{2} - \frac{1}{2} \]
\[ \left| X_2 \right| = \left( \left| X_1 \right| - 1 \right)/2 = \frac{n}{4} - \frac{3}{2} - \frac{3}{4} \]
\[ \vdots \]
\[ \left| X_i \right| \geq \frac{n}{2^i} - 1 \]

When \( i = 2(k-1) \),
\[ \left| X_i \right| \geq \frac{4^k}{2^{2(k-1)}} - 1 \geq 1 \]

we have a monox \( K_k \)!
Set colouring Ramsey bounds

Upper bound (Erdős - Szekeres) $R_{r,d}(k) = (r/d)^{rk}$
Set colouring Ramsey bounds

Upper bound (Erdős - Szekeres): $R_{r,d}(k) = \left( \frac{r}{d} \right)^{rk}$

Proof:

\[ \begin{array}{c}
\text{\ldots} \\
\text{\ldots} \\
\text{\ldots} \\
\end{array} \]
Set colouring Ramsey bounds

upper bound (Erdős - Szekeres) \( R_{r,\geq}(k) = \left(\frac{r}{\geq}\right)^{rk} \) same idea!

proof:

Given a vertex \( v \) in \( X_{i-1} \):

majority color present in at least neighbours.
Set colouring Ramsey bounds

upper bound (Erdős - Szekeres) \( R_{r,\rho}(k) = \left(\frac{r}{\rho}\right)^{\frac{r}{k}} \) same idea!

proof:

Given a vertex \( v \) in \( X_{i-1} \):

majority color present in at least \( |X_{i-1}| - 1 \) neighbours.
Set colouring Ramsey bounds

upper bound (Erdős - Szekeres) \( R_{r,s}(k) = \left( \frac{r}{s} \right)^r k \) same idea!

proof:

Given a vertex \( v \) in \( X_{i-1} \):

majority color present in at least
\( \geq (1-\varepsilon i) \) neighbours.
Set colouring Ramsey bounds

upper bound (Erdős - Szekeres) $R_{r,s}(k) = \left\lfloor \frac{r^k}{s} \right\rfloor$ same idea!

proof:

Given a vertex $v$ in $X_{i-1}$:

majority color present in at least $\frac{\phi}{r}(lx_{i-1} - 1 - 1)$ neighbours.
Set colouring Ramsey bounds

**Upper bound (Erdős – Szekeres)** $R_{r,s}(k) = \left( \frac{r}{s} \right)^k$ same idea!

**Proof:**

Given a vertex $v$ in $X_{i-1}$:

- Majority color present in at least
  \[
  \frac{\Delta}{r} (l_{i-1} - 1) \text{ neighbours.}
  \]

\[
\Rightarrow l_{i-1} \geq \frac{\Delta}{r} (l_{i-1} - 1) \geq \left( \frac{\Delta}{r} \right)^i n - 1
\]
Set colouring Ramsey bounds

upper bound (Erdős - Szekeres)  $R_{r,s}(k) = \left(\frac{r}{s}\right)^{rk}$ same idea!

proof:

Given a vertex $v$ in $X_{i-1}$:

majority color present in at least

$$\frac{s}{r} (1x_{i-1} - 1)$$ neighbours.

$$\Rightarrow 1x_i \geq \frac{s}{r} (1x_{i-1} - 1) \geq \left(\frac{s}{r}\right)^i n - 1$$

At time $i = r(k-s)$ we have a monochromatic clique!
Set colouring Ramsey bounds

lower bound $R_{r,r-1}(k) = 2^{\sqrt{2^k}}$ (case $\alpha = r-1$)

proof:
Set colouring Ramsey bounds

Lower bound $R_{r,r-1}(k) = 2^\pi(k/r)$ (case $\delta = r - 1$)

Proof:
Assume $r \leq k$. 

For $r > k$: $R_{r,r-1}(k) = \min \{ n : (\mathbb{Z}) > r \}$

Consider a random colouring (ind. and unif. in $([r])$)
Set colouring Ramsey bounds

Lower bound \( R_{r, r-1}(k) = 2^{\Omega(k/r)} \) (case \( \beta = r-1 \))

Proof:
Assume \( r \leq k \).

For \( r > k \): \( R_{r, r-1}(k) = \min \{ n : \binom{\lfloor r/2 \rfloor}{2} > r \} \)

Consider a random colouring \( \text{(ind. and unif. in } \binom{2}{2} \text{)} \)

\( \text{P( fixed k-clique is monox }) \leq \binom{k}{2} \frac{\beta}{r} \)
Set colouring Ramsey bounds

Lower bound \( R_{r,r-1}(k) = 2^{\mathcal{O}(k/r)} \) (case \( \delta = r-1 \))

Proof:

Assume \( r \leq k \).

\[ r > k : \quad R_{r,r-1}(k) = \min \{ n : \binom{n}{2} > r \} \]

Consider a random colouring (ind. and unif. in \( \binom{\mathcal{S}(n)}{2} \))

\[ \mathbb{P}(\text{fixed } k\text{-clique is monoch.}) \leq r \left( \frac{\mathcal{S}(n)}{r} \right)^{\binom{k}{2}} \]

\[ \mathbb{E}[\# \text{ of monoch. } k\text{-clique}] \leq \binom{\mathcal{S}(n)}{r} r \left( 1 - 1/r \right)^{\binom{k}{2}} \]
Set colouring Ramsey bounds

lower bound \( R_{r,r-1}(k) = 2^{\frac{r}{k}}(k/r) \) (case \( \delta = r-1 \))

proof:

Assume \( r \leq k \).

\( r > k : R_{r,r-1}(k) = \min \{ n : \left( \frac{2}{r} \right)^n > r \} \)

consider a random colouring (ind. and unif. in \( \binom{[r]}{2} \))

\( \Pr( \text{fixed } k\text{-clique is mono}x ) \leq r \left( \frac{2}{r} \right)^{\frac{k}{2}} \)

\( E \left[ \# \text{ of mono}x k\text{-clique} \right] \leq \binom{n}{k} r (1 - 1/r)^{\left( \frac{k}{2} \right)} \)

\( \leq r n^k \left( 1 - 1/r \right)^{\frac{k}{2}} \leq (n^{1/k} e^{-k/r})^k \)

\( n = 2^{\frac{r}{k}}(k/r) \quad < 1 \)
Set colouring Ramsey

Originally studied by Erdős, Hajnal and Rado (65)

Conjectured \( R_{r, r-\delta} (k) \leq 2^{\delta r} k \) for some \( \delta (r) \rightarrow 0 \) as \( n \rightarrow \infty \)
Set colouring Ramsey

L. originally studied by Erdős, Hajnal and Rado (65)

L. conjectured \( R_{r,r-\delta}(k) \leq 2^{\delta(r)k} \) for some \( \delta(r) \to 0 \) as \( n \to \infty \)

Erdős and Szemerédi (72)

\[ 2^{o(k/r)} \leq R_{r,r-\delta}(k) \leq o(k/r) \]
Set colouring Ramsey

Lo originally studied by Erdős, Hajnal and Rado (65)

Lo conjectured $R_{r,r-3}(k) \leq 2^{8(r)k}$ for some $8(r) \to 0$ as $n \to \infty$

Erdős and Szemeredi (72)

$$2^{\Omega(k/r)} \leq R_{r,r-3}(k) \leq r^{O(k/r)}$$

"Simple bounds"

$$2^{\Omega(k/r)} \leq R_{r,r-3}(k) \leq (r/r-3)^r k$$
Set colouring Ramsey

general setting:

"simple bounds"

\[ 2^{\frac{2}{k} \left( k \left( r - \delta \right) / r \right)} \leq R_{r, \delta} (k) \leq \left( \frac{r}{\delta} \right)^{r k} \]
Set colouring Ramsey

general setting:

"simple bounds"

\[ 2^{\frac{\log (k(r-a)/r)}{r}} \leq R_{r,s}(k) \leq (r/s)^{rk} \]

Conlon, Fox, He, Mubayi, Suk and Verstraëte (22+)

\[ R_{r,s}(k) = 2^\Theta(kr) \]

for every \( s = o(r) \) s.t. \( s/r \) is bounded away from 0 and 1
Set colouring Ramsey

Conlon, Fox, He, Mubayi, Suk and Verstraëte (22+)

\[ R_{r, \delta}(k) = 2^{\Theta(kr)} \]

for every \( \delta = \delta(r) \) s.t. \( \delta/r \) is bounded away from 0 and 1.
Set colouring Ramsey

Conlon, Fox, He, Mubayi, Suk and Verstraëte (22+)

\[ R_{r,n}(k) = 2^{\Theta(kr)} \]

for every \( d = o(r) \) s.t. \( d/r \) is bounded away from 0 and 1 (u.b. and l.b. exponents differ by \( O(\log r) \))

Lower and upper bounds diverge significantly when

\( d = o(r) \) or \( r - d = o(r) \)
Set colouring Ramsey

Conlon, Fox, He, Mubayi, Suk and Verstraëte (22+)

\[ R_{r,s}(k) = 2^{\Theta(kr)} \] — best upper bound known!

For every \( s = o(r) \) s.t. \( s/r \) is bounded away from 0 and 1 (u.b. and l.b. exponents differ by \( O(\log r) \))

Lower and upper bounds diverge significantly when \( s = \omega(r) \) or \( r-s = o(r) \)

Ex.: \( 2^{ck/\sqrt{r}} \leq R_{r,r-\sqrt{r}}(k) \leq 2^{c'k \log r} \)
Set colouring Ramsey

Conlon, Fox, He, Mubayi, Suk and Verstraëte (22+)

\[ R_{r,n}(k) = 2^\Theta(kr) \]

for every \( d = o(r) \) s.t. \( d/r \) is bounded away from 0 and 1

Aragão, Collares, Marciano, M., Morris

\[ \exists c, \delta > 0 \text{ s.t. if } d < r - c \log r, \text{ then} \]

\[ R_{r,\delta}(k) \geq \exp(\delta \varepsilon^2 r k) \]

\( \forall k > (c/\delta) \log r \), where \( \varepsilon = (r-\delta)/r \).
Set colouring Ramsey

Conlon, Fox, He, Mubayi, Suk and Verstraëte (22+)

\[ R_{r,\alpha}(k) = 2^{\Theta(kr)} \]

for every \( \alpha = \delta(r) \) s.t. \( \alpha / r \) is bounded away from 0 and 1.

Aragão, Collares, Marciano, M., Morris

\[ \exists c, \delta > 0 \text{ s.t. if } \alpha < r - c \log r, \text{ then } \]

\[ R_{r,\alpha}(k) > \exp(\delta \epsilon^2 r k) \]

\[ \forall k > (1+\delta) / \epsilon + 1 \]

\[ \forall k > (c / \epsilon) \log r, \text{ where } \epsilon = (r - \alpha) / r. \]
Set colouring Ramsey

\[ R_{r, \delta}(k) = 2^{\tilde{\Theta}(\epsilon^2 r k)} \]

Remark:

<table>
<thead>
<tr>
<th>( r - \delta = \Omega(r) )</th>
<th>L.B.</th>
<th>( \exp. \Delta \to U.B. )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta = r - o(r) )</td>
<td>CFHMSV</td>
<td>( O(\log r) )</td>
</tr>
<tr>
<td>and ( \delta &gt; r - c \log r )</td>
<td>simple bound</td>
<td>( O((\log r)^2) )</td>
</tr>
<tr>
<td>( \delta \leq r - c \log r )</td>
<td>ACMMM</td>
<td>( O(\log r) )</td>
</tr>
</tbody>
</table>
Lower bound construction

\[ s \leq r - c \log r, \quad n = \exp (\delta e^2 r k) \]

\text{Lwts:} \quad \exists \text{ a colouring } c : E(K_n) \to \left[ \frac{[r]}{s} \right] \text{ with no } K_k \text{ mono.}

\text{proof:}
Lower bound construction

\[ r \leq r - c \log r, \quad n = \exp \left( s e^2 r k \right) \]

Let WTS: \exists a colouring \( c: E(K_n) \to \left( \{r\} \right) \) with no \( K_k \) mono x.

**Proof:**

For each color \( i \in \{r\} \) consider

\[ H_i = G(m, p) \quad \text{and} \quad \phi_i: [n] \to [m] \text{ ind. } \text{unif}. \]
Lower bound construction

\[ \sigma \leq r - c \log r, \quad n = \exp(5e^2r_k) \]

Let \( \exists \) a colouring \( c : E(K_n) \to (\mathbb{Z}_\sigma) \) with no \( K_k \) mono-\( x \).

**Proof:**

For each color \( i \in \mathbb{Z}_\sigma \) consider

\[ H_i = G(m, p) \quad \text{and} \quad \phi_i : [n] \to [m] \text{ ind. unif.} \]
Lower bound construction

\[ n = \exp(\delta e^2 r k) \]

\[ \exists \text{ wts: } \exists \text{ a colouring } c : E(K_n) \rightarrow ([r]) \text{ with no } K_k \text{ monox.} \]

\text{proof:}

For each color \( i \in [r] \) consider

\[ H_i = G(m, p) \quad \text{and} \quad \phi_i : [n] \rightarrow [m] \text{ ind. unif.} \]
Lower bound construction

WTS: \exists a colouring \ c: E(K_n) \to ([\mathcal{S}]) \text{ with no } K_k \text{ mono } x.

H_i = G(m, p) \text{ and } \phi_i: [n] \to [m] \text{ ind. unif.}
Lower bound construction

\[ \exists \text{ a colouring } c : E(k_n) \rightarrow \binom{[r]}{2} \text{ with no } K_k \text{ monochromatic.} \]

\[ H_i = G(m,p) \quad \text{and} \quad \phi_i : [n] \rightarrow [m] \text{ ind. unif.} \]

\[ \chi'(uv) = \{ i \in [r] : \{ \phi_i(u), \phi_i(v) \} \in E(H_i) \} \quad \forall \ uv \in E(k_n) \]

Bad edges: \[ B = \{ e \in E(k_n) : |\chi'(e)| < r \} \]
Lower bound construction

\[ \text{wts: } \exists \text{ a colouring } c : E(k_n) \to \left( \frac{[r]}{2} \right) \text{ with no } K_k \text{ monoex.} \]

\[ H_i = G(m,p) \text{ and } \phi_i : [n] \to [m] \text{ ind. unif.} \]

\[ \chi'(uv) = \{ i \in [r] : \{ \phi_i(u), \phi_i(v) \} \in E(H_i) \} \quad \forall uv \in E(k_n) \]

Bad edges: \[ B = \{ e \in E(k_n) : |\chi'(e)| < r \} \]

uv-crossing colours: \[ K(uv) = \{ i \in [r] : \phi_i(u) \neq \phi_i(v) \} \quad j \in K(uv) \]

\[ \phi_i : [n] \to [m] \text{ ind. unif.} \]
Lower bound construction

\[ \text{wts: } \exists \text{ a colouring } c: E(K_n) \to [r] \text{ with no } K_k \text{ mono x.} \]

\[ H_i = G(m,p) \text{ and } \phi_i : [n] \to [m] \text{ ind. unif.} \]

\[ \chi'(uv) = \{ i \in [r] : \{ \phi_i(u), \phi_i(v) \} \in E(H_i) \} \forall uv \in E(K_n) \]

\[ \text{Bad edges: } B = \{ e \in E(K_n) : |\chi'(e)| < r \} \]

\[ \text{Crossing colours: } K(uv) = \{ i \in [r] : \phi_i(u) \neq \phi_i(v) \} \]

\[ j \in K(uv) \]

\[ 1 \in K(\omega \omega) \]
Lower bound construction

WTS: \( \exists \) a colouring \( c : E(K_n) \rightarrow \{1, \ldots, c\} \) with no \( K_k \) mono..x.

\[ H_i = G(m, p) \quad \text{and} \quad \phi_i : [n] \rightarrow [m] \text{ ind. unif.} \]

\[
\chi'(uv) = \{ i \in [r] : \{ \phi_i(u), \phi_i(v) \} \in E(H_i) \} \quad \forall uv \in E(K_n)
\]

Bad edges: \( B = \{ e \in E(K_n) : |\chi'(e)| < r \} \)

Crossing colours: \( K(uv) = \{ i \in [r] : \phi_i(u) \neq \phi_i(v) \} \)

 Colouring:

\[
\chi(e) = \begin{cases} 
\chi'(e) & \text{if } e \not\in B \\
K(e) & \text{if } e \in B
\end{cases}
\]
Lower bound construction

Properties we want (choice of parameters)

"Almost all" edges of \( x \) have \( \geq \delta \) colours

\[ p > 1 - c \varepsilon \] (taking \( p \) small helps)

\( B \) is small
Lower bound construction

Properties we want (choice of parameters)

"Almost all" edges of $\mathcal{X}$ have $\gg \gg$ colours

$\Rightarrow p > 1 - ce$ (taking $p$ small helps)

Choose $m$ s.t.

every $K$-set misses $\mathcal{R}(\varepsilon k^2)$ edges in every colour class.

$\Rightarrow G(m, p)$ is unlikely to contain a $K_k$
Lower bound construction

Properties we want (choice of parameters)

Almost all edges of $\mathcal{K}$ have $> \delta$ colours

$\rightarrow B$ is small

Choose $m$ s.t.

every $\mathcal{E}$-set misses $\mathcal{E} \mathcal{K}^2$ edges in every colour class.

$\rightarrow G(m, p)$ is unlikely to contain a $\mathcal{K}_k$

"backup colouring" is s.t. no edge misses more than $c \mathcal{E}$ colours

$\rightarrow m = 2^{\delta^2 \mathcal{E} k/2}$ satisfies both!
Lower bound construction

\( \chi \) is a valid colouring!

• Every edge receives \( \geq 1 \) colours

\[ \text{by definition } + 1 k(e) \geq \forall e \in E(K_n) \quad (\text{first moment}) \]
Lower bound construction

X is a valid colouring!

- Every edge receives > colours
  
  $\mu \text{ definition } + 1k(e) \geq \forall e \in E(K_n)$ (first moment)

- Whp no $K_k$ mono $X$.

  $\mu$ cliques with few bad edges (Chernoff)

  $\mu$ cliques with many bad edges ($\geq 5\epsilon k^2$)

  $\mu$ technical part

  need to use randomness of the partitions $\phi_i$'s
Open problems

• reduce the $(\log r)^2$ gap when $r - s = c \log r$ and $\varepsilon k = \log r$.

• hypergraph set colouring Ramsey number

Lo problems in CFHMSV:

prove $R^{(3)}_{5,2}(k)$ is double exponential in $n$. 
Open problems

- reduce the \((\log r)^2\) gap when \(r - s = C \log r\) and \(\varepsilon k = \log r\).
- hypergraph set colouring Ramsey number

Lo problems in CFHMSV:

prove \(R^{(3)}_{5,2}(k)\) is double exponential in \(n\)

Thank you!
Set colouring Ramsey bounds

upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r,\Delta}(k) \leq 2^{\Theta(kr)} \]

sketch:

Traditional Erdős - Szekeres:

- neighbourhood chasing argument
- keeping track of size of mono\(\times\) cliques
Set colouring Ramsey bounds

Upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

$$R_{r,2}(k) < 2^\Theta(kr)$$

Sketch:

Modified Erdős-Szekeres:

- Neighbourhood chasing argument
- Only in active colours
- Keeping track of size of monochromatic cliques
- Lo for active colours
Set colouring Ramsey bounds

Upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r, \odot} (k) < 2^{\Theta(kr)} \]

Sketch:

\{ \text{no active colours} \}

\[ X_0 \]
Set colouring Ramsey bounds

upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r,d}(k) < 2^{\Theta(kr)} \]

Sketch:

- no active colours
- choose densest colour in \( x_0 \) to be activated (blue)
- reasonably dense in blue
Set colouring Ramsey bounds

upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r,\omega}(k) < 2^{\Theta(kr)} \]

sketch:

\[ \begin{cases} \text{no active colours} \\ \text{choose densest colour in } x_0 \text{ to be activated (blue)} \\ \text{reasonably dense in blue} \end{cases} \]
Set colouring Ramsey bounds

upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r,\Delta}(k) < 2^{\Theta(kr)} \]

}\} active colours

\[ \chi_1 \]
Set colouring Ramsey bounds

upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r,\infty}(k) < 2^{\Theta(kr)} \]

Do the densities of all active colours are below \( \frac{1}{2} \) in \( x_i \)?

\( \exists \) yes, activate new colour.
Set colouring Ramsey bounds

upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r, \Delta}(k) < 2^{\Theta(kr)} \]

Do the densities of all active colours are below \( \frac{1}{2} \) in \( x_i \)?

\- Yes, activate new colour.
\- No! density of red \( \geq \frac{1}{2} \)

\[ |N_r(v)| \geq \frac{1}{2}x_i \]
Set colouring Ramsey bounds

upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r,\frac{r}{d}}(k) < 2^\Theta(kr) \]

Do the densities of all active colours are below \( 1/2 \) in \( x_i \)?

- Yes, activate new colour.
- No! Density of red \( > 1/2 \) \( |N_R(v)| > |x_i|/2 \).
Set colouring Ramsey bounds

Upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r, d}(k) < 2 \Theta(kr) \]

Do the densities of all active colours are below \( \frac{1}{2} \) in \( X_i \)?

\[ \text{If yes, activate new colour.} \]

\[ \text{If no! density of red } \geq \frac{1}{2} \]

\[ |N_r(v)| > |x_i| \frac{1}{2} \]
Set colouring Ramsey bounds

Upper bound (Conlon, Fox, He, Mubayi, Suk and Verstraëte)

\[ R_{r,\theta}(k) < 2^{\Theta(kr)} \]

active colours

Do the densities of all active colours are below \( \frac{1}{2} \) in \( X_i \)?

- Yes, activate new colour.
- No! Density of red \( > \frac{1}{2} \)
  \[ |N(v)_{\mathcal{R}}| > |X_i| \frac{1}{2} \]