Law of large numbers for ballistic random walks in dynamic random environments under lateral decoupling

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Random walks on random environments in $\mathbb{Z}$

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- Example:

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• Dynamic random environments given by a stochastic particle systems.
• Exclusion process, contact process, systems of independent random walks, etc.
• Laws of large numbers, central limit theorems, large deviation estimates, among others.
Our model

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  - $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.
  - $(E, \mathcal{E})$ a measurable space.
  - $\eta = (\eta_t)_{t \in \mathbb{R}^+}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ where $\eta_t = (\eta_t(x))_{x \in Z \in E}$ for each $t \in \mathbb{R}^+$. We assume that $\eta$ invariant by space-time translations.
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Our model

- Assumption DEC (Lateral Decoupling)

  \( f_1 \) and \( f_2 \) two non-negative functions, \( \| f_1 \|_\infty \leq 1, \| f_2 \|_\infty \leq 1 \).
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  \(f_i\) supported on \(B_i\).
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If \( d_H \geq v \circ d_V + C_1 s + C_2 \), \( v \circ, C_1, C_2 \), positive constants,
Assumption DEC (Lateral Decoupling)

$f_1$ and $f_2$ two non-negative functions, $\|f_1\|_\infty \leq 1$, $\|f_2\|_\infty \leq 1$. $f_i$ supported on $B_i$.

If $d_H \geq \nu d_V + C_1 s + C_2$, $\nu$, $C_1$, $C_2$, positive constants, then
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  If \( d_H \geq v_\circ d_V + C_1 s + C_2 \), \( v_\circ, C_1, C_2 \), positive constants, then

  \[ \mathbb{E}[f_1 f_2] \leq \mathbb{E}[f_1] \mathbb{E}[f_2] + C_\circ e^{-\kappa_\circ (\log d_H)^{\gamma_\circ}}, \quad C_\circ, \kappa_\circ \in \mathbb{R}_+, \gamma_\circ > 1. \]
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- \( \alpha, \beta : E \rightarrow \mathbb{R}_+ \).
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- \( X_0 = 0 \).
- If \( X_t = x \), then \( X \) jumps to the right with rate \( \alpha(\eta_t(x)) \) and jumps to the left with rate \( \beta(\eta_t(x)) \).
Our model

The random walk

- $\alpha, \beta : E \rightarrow \mathbb{R}_+$. 
- Continuous-time random walk $X = (X_t)_{t \geq 0}$. 
- $X_0 = 0$. 
- If $X_t = x$, then $X$ jumps to the right with rate $\alpha(\eta_t(x))$ and jumps to the left with rate $\beta(\eta_t(x))$. 

\[ \sup_{\xi \in E} \{\alpha(\xi) + \beta(\xi)\} \leq \Lambda. \]
Assumption BAL (Ballisticity)
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There exist constants $\nu_\star, \kappa_\star, C_\star > 0$ and $\gamma_\star > 1$ such that

$$\mathbb{P}(X_t \leq \nu_\star t) \leq C_\star e^{-\kappa_\star \log t} \gamma_\star$$

for all $t \geq 0$. 
Main theorem

**Theorem**

Assume that Assumptions DEC and BAL hold with $v_* > v_\circ$. Then there exists a speed $v \geq v_*$ such that

$$\lim_{t \to \infty} \frac{X_t}{t} = v \quad \mathbb{P}\text{-almost surely.}$$
Notations and definitions

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- \( H \geq 1, \, v \in \mathbb{R}, \, w \in \mathbb{R}^2 \)

\[
A_{H,w}(v) = \left[ \exists \, y \in I_H(w) \cap L \text{ s.t. } X_H^y - \pi_1(y) \geq vH \right],
\]

where \( I_H(w) = w + [0, \lambda H) \times \{0\} \).
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- \( H \geq 1, \; \nu \in \mathbb{R}, \; w \in \mathbb{R}^2 \)

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A_{H,w}(\nu) = \left\{ \exists \; y \in I_H(w) \cap \mathbb{L} \; \text{s.t.} \; X^y_H - \pi_1(y) \geq \nu H \right\},
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Notations and definitions

- \( p_H(v) := \sup_{w \in \mathbb{R}^2} \mathbb{P}[A_{H,w}(v)] \)

- \( v^- := \sup \{ v \in \mathbb{R} : \lim \inf_{H \to \infty} \tilde{p}_H(v) = 0 \} \)

- \( v^-, v^+ \in [v^\star, \lambda] \)

- Probability goes to zero
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- \( v_-, v_+ \in [v_*, \lambda] \)
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probability goes to zero

\( v_- \)

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positive probability

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Proposition 1

∀ε > 0, ∃c₁ = c₁(ε) > 0 such that

\[ p_H(v_+ + \varepsilon) \leq c₁ e^{-3\kappa \log \gamma H} \quad \text{and} \quad \tilde{p}_H(v_- - \varepsilon) \leq c₁ e^{-3\kappa \log \gamma H}, \]

for all \( H \geq 1 \).
Proposition 1

∀ε > 0, ∃c₁ = c₁(ε) > 0 such that

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- Proposition 1 implies that \( v_- \leq v_+ \).
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\[ \begin{array}{ccc}
    & v_+ & \\
\text{hard} & \downarrow & \text{hard} \\
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    & v_- \\
\end{array} \]
Proposition 1

\( \forall \varepsilon > 0, \exists c_1 = c_1(\varepsilon) > 0 \text{ such that} \)

\[ p_H(v_+ + \varepsilon) \leq c_1 e^{-3\kappa \log \gamma H} \quad \text{and} \quad \tilde{p}_H(v_- - \varepsilon) \leq c_1 e^{-3\kappa \log \gamma H}, \]

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Proposition

We have

\[ v_+ = v_- . \]
Proof of Proposition 1 - Multiscale renormalization

- Define \((L_k)_{k \geq 0}\) inductively as

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L_0 := 10^{10} \quad \text{and} \quad L_{k+1} := \ell_k L_k, \quad \text{for } k \geq 0,
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where \(\ell_k := \lfloor L_k^\nu \rfloor\) and \(\nu \in (0, 1)\) is chosen properly.

- \(h \geq 1\)

\[ B_{L_k}^h := [-4\lambda h L_k, 5\lambda h L_k] \times [0, h L_k) \subset \mathbb{R}^2, \]
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- For \(w \in \mathbb{R}^2\), we write

  \[
  B^h_{L_k}(w) := w + B^h_{L_k}
  \]

- \(m = (h, k, w)\)

  \[
  B_m := B^h_{L_k}(w), \quad A_m(v) := A_{hL_k, w}(v) \quad \text{and} \quad \tilde{A}_m(v) := \tilde{A}_{hL_k, w}(v).
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\[ y + hL_{k+1}(v_{k+1}, 1) \]

\[ 9\lambda hL_{k+1} \]

\[ B_{m_2} \]

\[ 5\lambda hL_k \]

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\[ hL_k \]

\[ d_H \]

\[ d_V \]
Proof of Proposition 1 - Multiscale renormalization

- $d_H > v \circ d_V + (\lambda + C_1 + C_2) hL_k \geq v \circ d_V + C_1 hL_k + C_2$

- The other case is similar
Proof of Proposition 1 - Multiscale renormalization

- $d_H > \nu \circ d_V + (\lambda + C_1 + C_2) hL_k \geq \nu \circ d_V + C_1 hL_k + C_2$

- $p_{hL_{k+1}} \leq c \ell_k^4 p_{hL_k}^2(v_k) + e^{-8\kappa (\log hL_k)^\gamma}$
Proof of Proposition 1 - Multiscale renormalization

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- \( p_{hL_{k+1}} \leq c\ell_k^4 p_{hL_k}(v_k) + e^{-8\kappa(\log hL_k)^\gamma} \)
- **Recurrence**

  if \( p_{hL_k}(v_k) \leq e^{-4\kappa \log^\gamma L_k} \) then \( p_{hL_{k+1}}(v_{k+1}) \leq e^{-4\kappa \log^\gamma L_{k+1}} \).
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  if $p_{hL_k}(v_k) \leq e^{-4\kappa \log^\gamma L_k}$ then $p_{hL_{k+1}}(v_{k+1}) \leq e^{-4\kappa \log^\gamma L_{k+1}}$.

- **Triggering:**

  $$p_{c_2 L_k} (v) \leq e^{-4\kappa \log^\gamma L_k} \quad \text{for all } k \geq k_0.$$
\begin{itemize}
\item $d_H > v \circ d_V + (\lambda + C_1 + C_2) hL_k \geq v \circ d_V + C_1 hL_k + C_2$
\item $p_{hL_{k+1}} \leq c \ell_k^4 p_{hL_k}^2(v_k) + e^{-8\kappa (\log hL_k)^\gamma}$
\item **Recurrence**

\[\text{if } p_{hL_k}(v_k) \leq e^{-4\kappa \log^\gamma L_k} \text{ then } p_{hL_{k+1}}(v_{k+1}) \leq e^{-4\kappa \log^\gamma L_{k+1}}.\]
\item **Triggering:**

\[p_{c_2 L_k}(v) \leq e^{-4\kappa \log^\gamma L_k} \text{ for all } k \geq k_0.\]
\item **Interpolation:** $\epsilon > 0, H \geq 1$
\end{itemize}
Proof of Proposition 1 - Multiscale renormalization

- \( d_H > v_0 d_V + (\lambda + C_1 + C_2) h L_k \geq v_0 d_V + C_1 h L_k + C_2 \)
- \( p_{h L_k + 1} \leq c \ell_k^4 p_{h L_k}^2 (v_k) + e^{-8 \kappa (\log h L_k)^\gamma} \)
- **Recurrence**

  if \( p_{h L_k} (v_k) \leq e^{-4 \kappa \log \gamma L_k} \) then \( p_{h L_k + 1} (v_{k+1}) \leq e^{-4 \kappa \log \gamma L_{k+1}} \).

- **Triggering:**

  \[ p_{c_2 L_k} (v) \leq e^{-4 \kappa \log \gamma L_k} \quad \text{for all } k \geq k_0. \]

- **Interpolation:** \( \epsilon > 0, \ H \geq 1 \)

  \[ p_H (v_+ + \epsilon) \leq c_1 e^{-3 \kappa \log \gamma H} \]
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- \( p_{hL_{k+1}} \leq c \ell_k^4 p_{hL_k}^2 (v_k) + e^{-8\kappa (\log hL_k)^\gamma} \)
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- The other case is similar
Zero range process

- Initial configuration $\eta_0$: 
**Zero range process**

- Initial configuration \( \eta_0 \):
- \( g : \mathbb{N}_0 \rightarrow \mathbb{R}_+ \) with \( g(0) = 0 \).
Example of environments that satisfy DEC

Zero range process

- Initial configuration $\eta_0$:
- $g : \mathbb{N}_0 \rightarrow \mathbb{R}_+ \text{ with } g(0) = 0$.
- $\Gamma_- \leq g(k) - g(k - 1) \leq \Gamma_+$, for all $k \geq 1$. 

Marginals: for $\phi \in \mathbb{R}_+$

$\nu_{\phi}(k) = \frac{1}{Z(\phi)} \phi^k g(k)!$, for all $k \in \mathbb{N}_0$ and $x \in \mathbb{Z}$.
Example of environments that satisfy DEC

Zero range process

- Initial configuration $\eta_0$:
- $g : \mathbb{N}_0 \to \mathbb{R}_+$ with $g(0) = 0$.
- $\Gamma_- \leq g(k) - g(k - 1) \leq \Gamma_+$, for all $k \geq 1$.
- Marginals: for $\phi \in \mathbb{R}_+$

$$
\nu_{\phi}(k) = \frac{1}{Z(\phi) g(k)!} \phi^k, \quad \text{for all } k \in \mathbb{N}_0 \text{ and } x \in \mathbb{Z}.
$$
The zero-range process in $\mathbb{Z}$

- Then, the initial configuration is given by the product measure of these marginals, which is invariant for the zero-range process.
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Evolution:

Figure: The evolution of the zero-range process.
- $f_1$ and $f_2$ two non-negative functions, $\|f_1\|_\infty \leq 1$, $\|f_2\|_\infty \leq 1$. 
- $f_1$ and $f_2$ two non-negative functions, $\|f_1\|_\infty \leq 1$, $\|f_2\|_\infty \leq 1$.
- $f_i$ supported on $B_i$. 

Other example: Asymmetric exclusion process
- $f_1$ and $f_2$ two non-negative functions, $\|f_1\|_\infty \leq 1$, $\|f_2\|_\infty \leq 1$.
- $f_i$ supported on $B_i$. 

\begin{itemize}
  \item \[d_H \geq C_1 (s + d_V) + C_2.\]
  \item \[E \rho [f_1 f_2] \leq E \rho [f_1] E \rho [f_2] + c_1 e^{-c_1^{-1} \log 5/4 d_H}, \gamma \circ = 5/4.\]
  \item Other example: Asymmetric exclusion process.
\begin{itemize}
\item $f_1$ and $f_2$ two non-negative functions, $\|f_1\|_{\infty} \leq 1$, $\|f_2\|_{\infty} \leq 1$.
\item $f_i$ supported on $B_i$.
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\end{itemize}
- $f_1$ and $f_2$ two non-negative functions, $\|f_1\|_\infty \leq 1$, $\|f_2\|_\infty \leq 1$.
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- $d_H \geq C_1(s + d_V) + C_2$.
- $\mathbb{E}^\rho[f_1 f_2] \leq \mathbb{E}^\rho[f_1] \mathbb{E}^\rho[f_2] + c_1 e^{-c_1^{-1} \log^{5/4} d_H}$, $\gamma_o = 5/4$.
- Other example: Asymmetric exclusion process
Future works

- Law of large numbers
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- CLT
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- higher dimensions
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Thank You!
References


