Survival of Renewal Contact Process

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Probability Seminar - IM - UFRJ

August, 14 - 2023
Summary

1. Contact Process (Classic).
   - Model description
   - Critical parameter

2. Renewal Contact Process.
   - Model description
   - Some important results

3. Some proof ideas

4. References
• This is a model to describe the spread of infectious diseases introduced by Harris in 1974.

• Consider a connected graph $G = (V, E)$. For example, $V = \mathbb{Z}$ or $\mathbb{Z}^d$ and $E$ equal to the set of nearest neighbors in $\mathbb{Z}$ or $\mathbb{Z}^d$.

• The vertices of $V$ represent individuals that can be healthy or infected.

• Begin with some infected individuals (usually only the origin) at time $t = 0$.

• Infected individuals become healthy at a rate of 1.

• Healthy individuals become sick at a rate equal to a given parameter $\lambda$ times the number of infected neighbors.

• The state of the population at time $t$ is described by a configuration $\xi_t \in \{0, 1\}^V$, where $\xi_t(x) = 0$ means that the individual $x$ is healthy at time $t$ and $\xi_t(x) = 1$ means that the individual $x$ is infected at time $t$. 
Contact Process - Model description

Figure: Graphical Representation of the Contact Process
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• Depending on the position of the cure times and the transmission times, the infection can be extinct or not. Higher values of $\lambda$ favor the survival of the infection.

• Critical parameter:

$$\lambda_c := \inf\{\lambda : P(\tau^{0} = \infty) > 0\},$$

where $\tau^{0} := \inf\{t : \xi_t^{0} \equiv 0\}$ and $\xi_t^{0}$ denotes the configuration of the Contact Process at time $t$ when only the origin is infected at time 0.

• If $\lambda < \lambda_c$, the Contact Process is subcritical.

• If $\lambda > \lambda_c$, the Contact Process is supercritical.

• Is $0 < \lambda_c(\mathbb{Z}^d) < \infty$?
Initial bounds for $\lambda_c$:

- $\lambda_c(\mathbb{Z}^d) \geq \frac{1}{2d}$ (comparison with branching process).
- $\lambda_c(\mathbb{Z}) \leq 2$ (Holley and Liggett in 1978).
- $\lambda_c(\mathbb{Z}^d) \leq \frac{2}{d}$ (comparison with a Contact Process in $\mathbb{Z}$ with parameter $\lambda d$).

Best bounds for $\lambda_c(\mathbb{Z})$ in the literature so far:

- $\lambda_c(\mathbb{Z}) > 1,539$ (Grillenberger and Ziezold in 1988).
- $\lambda_c(\mathbb{Z}) \leq 1,942$ (Liggett in 1995).
Some other results and facts about the critical parameter:

- If $d_1 < d_2$, then $\lambda_c(\mathbb{Z}^{d_1}) \geq \lambda_c(\mathbb{Z}^{d_2})$.

- If $V$ is finite, then the infection will be extinct with probability one for any value of $\lambda$.

- If $P_\lambda(\tau^0 = \infty) = 0$, then $P_\lambda(\tau^A = \infty) = 0$ for any $A \subset V$ with $|A| < \infty$.

- The infection in $\mathbb{Z}^d$ is extinct with probability one when $\lambda = \lambda_c(\mathbb{Z}^d)$ (Bezuidenhout and Grimmett in 1990)
Renewal Contact Process - Model Description

• Consider $V = \mathbb{Z}^d$ and $\mathcal{E}$ equal to the set of nearest neighbors in $\mathbb{Z}^d$.

• The Renewal Contact Process (RCP) extends the Classical Contact Process by allowing the cure times to be determined according to a more general renewal process. Introduced by Fontes, Marchetti, Mountford and Vares in 2019.

• Given a probability measure $\mu$ on $[0, \infty)$, let $\{T_{x,i}\}_{x \in \mathbb{Z}^d, i \in \mathbb{N}}$ be i.i.d. random variables with distribution $\mu$ and $S_{x,n} = T_{x,1} + \ldots + T_{x,n}$. The times $S_{x,n}$ will be the position of the cure marks in the RCP.

• Transmission attempts work in the same way as in the Classical Contact Process, i.e, healthy individuals become sick at a rate equal to a given parameter $\lambda$ times the number of infected neighbors.

• The state of the population at time $t$ is described by a configuration $\xi_t \in \{0, 1\}^{\mathbb{Z}^d}$, where $\xi_t(x) = 0$ means that the individual $x$ is healthy at time $t$ and $\xi_t(x) = 1$ means that the individual $x$ is infected at time $t$. 
Renewal Contact Process - Model Description

Advantage:

- More flexible models

Disadvantage:

- The process is no longer Markovian

Important problems:

- Identify sufficient conditions on $\mu$ to have $\lambda_c(\mu) = 0$ or to have $\lambda_c(\mu) > 0$
- Identify sufficient conditions on $\mu$ to have $\lambda_c(\mu) < \infty$
Theorem 1: [Fontes, Marchetti, Mountford, Vares (2019)]

*If the distribution $\mu$ satisfies $\mu([t, \infty)) > t^{-\alpha}$ for all large $t$ and some $\alpha < 1$ (plus some regularity conditions), then the RCP($\mu$) survives with positive probability for any $\lambda > 0$ (valid for any $d \geq 1$)*
Theorem 2: [Fontes, Mountford, Vares (2020)]

If the distribution $\mu$ satisfies
\[ \int t^2 \mu(dt) < \infty, \]
then the RCP($\mu$) has $\lambda_c(\mu) > 0$ (valid for any $d \geq 1$)

Theorem 3: [Fontes, Mountford, Vares (2020)]

If $\mu$ has density $f$ and distribution function $F$ satisfying

(i) $\int t^\alpha \mu(dt) < \infty$, for some $\alpha > 1$;

(ii) the hazard rate $\frac{f(t)}{1 - F(t)}$ is decreasing in $t$,

then the RCP($\mu$) has $\lambda_c(\mu) > 0$ (valid only for $d = 1$)
Theorem 4: [Fontes, Mountford, Ungaretti, Vares (2023)]

If the distribution $\mu$ satisfies

$$\int_{1}^{\infty} t \exp \left[ \theta \sqrt{\log(t)} \right] \mu(dt) < \infty \quad \text{for some} \quad \theta > 4 \sqrt{d \log(2)},$$

then the RCP($\mu$) has $\lambda_c(\mu) > 0$. In particular, $\lambda_c(\mu) > 0$ whenever $\int t^\alpha \mu(dt) < \infty$ for some $\alpha > 1$ (valid for any $d \geq 1$).
So, we already have some results regarding conditions on $\mu$ to have $\lambda_c = 0$ or $\lambda_c > 0$. But what about conditions on $\mu$ to have $\lambda_c < \infty$ or $\lambda_c = \infty$?
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If $\mu$ has a density and a hazard rate that is bounded and decreasing, then $\lambda_c(\mu) < \infty$ (Fontes, Mountford, Ungaretti and Vares (2023)).
Theorem 5: [Santos, Vares (to appear)]

If $\mu$ is a continuous distribution satisfying $\mu[0, b] = 1$ for some $b < \infty$, then $\lambda_c(\mu) < \infty$ (valid for any $d \geq 1$).

Theorem 6: [Santos, Vares (to appear)]

If $\mu$ is an absolutely continuous distribution on $[0, \infty)$ with density $f$ and distribution function $F$ such that its hazard rate $\frac{f(t)}{1-F(t)}$ is decreasing in $t$, then $\lambda_c(\mu) < \infty$ (valid for any $d \geq 1$).
- Both proofs involve coupling the RCP(\(\mu\)) with supercritical oriented percolation.

- We create some convenient vertical and horizontal boxes in \(\mathbb{Z}^+ \times \mathbb{R}^+\). The boxes will be classified as “good” or “bad” depending if there is a path that allows the infection to cross the box in the desired direction.

- These boxes will be chosen in such a way that if we can find an infinite path constructed only using good boxes, then the RCP(\(\mu\)) survives.

- For any fixed \(\epsilon > 0\), we can find \(\tilde{\lambda}(\epsilon) < \infty\) such that the probability of any box (vertical or horizontal) to be good is greater than \(1 - \epsilon\) if we take \(\lambda \geq \tilde{\lambda}(\epsilon)\).

- Events involving boxes with edges in the same vertical line are not independent. We deal with this problem in each Theorem with a different approach.
Figure: Event “The box is good”
Contact Process - References


Renewal Contact Process - References


