Survival of Renewal Contact Process

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Summary

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2. Renewal Contact Process.
   - Model description
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Contact Process - Model description

- This is a model to describe the spread of infectious diseases introduced by Harris in 1974.
- Consider a connected graph $G = (V, \mathcal{E})$. For example, $V = \mathbb{Z}$ or $\mathbb{Z}^d$ and $\mathcal{E}$ equal to the set of nearest neighbors in $\mathbb{Z}$ or $\mathbb{Z}^d$.
- The vertices of V represent individuals that can be healthy or infected.
- Begin with some infected individuals (usually only the origin) at time $t = 0$.
- Infected individuals become healthy at a rate of 1.
- Healthy individuals become sick at a rate equal to a given parameter λ times the number of infected neighbors.
- The state of the population at time t is described by a configuration $\xi_t \in \{0,1\}^V$, where $\xi_t(x) = 0$ means that the individual x is healthy at time t and $\xi_t(x) = 1$ means that the individual x is infected at time t.

Contact Process - Model description

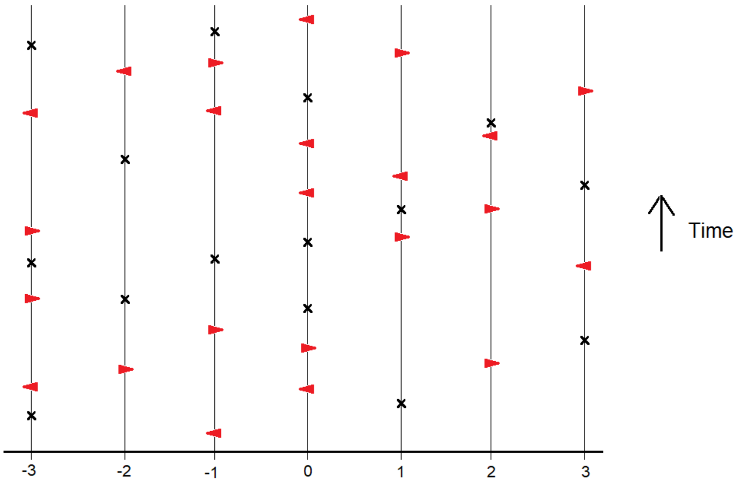
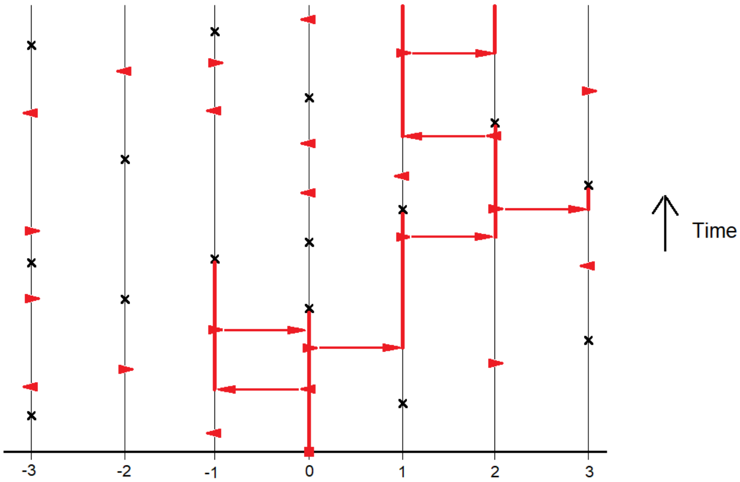


Figure: Graphical Representation of the Contact Process

Contact Process - Model description



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Contact Process - Model description

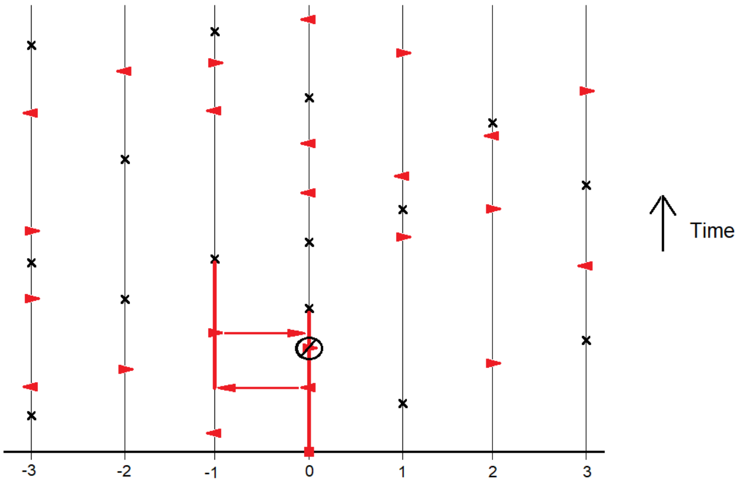

Figure: Graphical Representation of the Contact Process

Contact Process - Critical Parameter

- Depending on the position of the cure times and the transmission times, the infection can be extinct or not. Higher values of λ favor the survival of the infection.

- Critical parameter:

$$\lambda_c := \inf\{\lambda : P(\tau^{\{0\}} = \infty) > 0\},$$

where $\tau^{\{0\}} := \inf\{t : \xi_t^{\{0\}} \equiv 0\}$ and $\xi_t^{\{0\}}$ denotes the configuration of the Contact Process at time t when only the origin is infected at time 0.

- If $\lambda < \lambda_c$, the Contact Process is subcritical.
- If $\lambda > \lambda_c$, the Contact Process is supercritical.

- Is $0 < \lambda_c(\mathbb{Z}^d) < \infty$?

Contact Process - Critical Parameter

Initial bounds for λ_c:

- $\lambda_c(\mathbb{Z}^d) \geq \frac{1}{2d}$ (comparison with branching process).
- $\lambda_c(\mathbb{Z}) \leq 2$ (Holley and Liggett in 1978).
- $\lambda_c(\mathbb{Z}^d) \leq \frac{2}{d}$ (comparison with a Contact Process in $\mathbb{Z}$ with parameter λd).

Best bounds for $\lambda_c(\mathbb{Z})$ in the literature so far:

- $\lambda_c(\mathbb{Z}) > 1,539$ (Grillenberger and Ziezold in 1988).
- $\lambda_c(\mathbb{Z}) \leq 1,942$ (Liggett in 1995).

Contact Process - Critical Parameter

Some other results and facts about the critical parameter:

- If $d_1 < d_2$, then $\lambda_c(\mathbb{Z}^{d_1}) \geq \lambda_c(\mathbb{Z}^{d_2})$.
- If V is finite, then the infection will be extinct with probability one for any value of λ.
- If $P_\lambda(\tau^{\{0\}} = \infty) = 0$, then $P_\lambda(\tau^{\{A\}} = \infty) = 0$ for any $A \subset V$ with $|A| < \infty$.
- The infection in $\mathbb{Z}^d$ is extinct with probability one when $\lambda = \lambda_c(\mathbb{Z}^d)$ (Bezuidenhout and Grimmett in 1990)

Renewal Contact Process - Model Description

- Consider $V = \mathbb{Z}^d$ and $\mathcal{E}$ equal to the set of nearest neighbors in $\mathbb{Z}^d$.
- The Renewal Contact Process (RCP) extends the Classical Contact Process by allowing the cure times to be determined according to a more general renewal process. Introduced by Fontes, Marchetti, Mountford and Vares in 2019.
- Given a probability measure μ on $[0, \infty)$, let $\{T_{x,i}\}_{x \in \mathbb{Z}^d, i \in \mathbb{N}}$ be i.i.d. random variables with distribution μ and $S_{x,n} = T_{x,1} + \ldots + T_{x,n}$. The times $S_{x,n}$ will be the position of the cure marks in the RCP.
- Transmission attempts work in the same way as in the Classical Contact Process, i.e, healthy individuals become sick at a rate equal to a given parameter λ times the number of infected neighbors.
- The state of the population at time t is described by a configuration $\xi_t \in \{0, 1\}^{\mathbb{Z}^d}$, where $\xi_t(x) = 0$ means that the individual x is healthy at time t and $\xi_t(x) = 1$ means that the individual x is infected at time t.

Renewal Contact Process - Model Description

Advantage:

- More flexible models

Disadvantage:

- The process is no longer Markovian

Important problems:

- Identify sufficient conditions on μ to have $\lambda_c(\mu) = 0$ or to have $\lambda_c(\mu) > 0$
- Identify sufficient conditions on μ to have $\lambda_c(\mu) < \infty$

Renewal Contact Process - Some important results

Theorem 1: [Fontes, Marchetti, Mountford, Vares (2019)]

If the distribution μ *satisfies* $\mu([t,\infty)) > t^{-\alpha}$ *for all large* t *and some* $\alpha < 1$ *(plus some regularity conditions), then the RCP(*μ*) survives with positive probability for any* $\lambda > 0$ *(valid for any* $d \geq 1$*)*

Renewal Contact Process - Some important results

Theorem 2: [Fontes, Mountford, Vares (2020)]

If the distribution μ satisfies

$$\int t^2 \mu(dt) < \infty,$$

then the RCP(μ) has $\lambda_c(\mu) > 0$ (valid for any $d \geq 1$)

Theorem 3: [Fontes, Mountford, Vares (2020)]

If μ has density f and distribution function F satisfying

$$(i) \int t^\alpha \mu(dt) < \infty, \text{ for some } \alpha > 1;$$

$$(ii) \text{ the hazard rate } \frac{f(t)}{1 - F(t)} \text{ is decreasing in } t,$$

then the RCP(μ) has $\lambda_c(\mu) > 0$ (valid only for $d = 1$)

Theorem 4: [Fontes, Mountford, Ungaretti, Vares (2023)]

If the distribution μ satisfies

$$\int_1^\infty t \exp\left[\theta\sqrt{\log(t)}\right]\mu(dt) < \infty \quad \textit{for some} \quad \theta > 4\sqrt{d\log(2)},$$

then the RCP(μ) has $\lambda_c(\mu) > 0$. In particular, $\lambda_c(\mu) > 0$ whenever $\int t^\alpha \mu(dt) < \infty$ for some $\alpha > 1$ (valid for any $d \geq 1$).

So, we already have some results regarding conditions on μ to have $\lambda_c = 0$ or $\lambda_c > 0$. But what about conditions on μ to have $\lambda_c < \infty$ or $\lambda_c = \infty$?

So, we already have some results regarding conditions on μ to have $\lambda_c = 0$ or $\lambda_c > 0$. But what about conditions on μ to have $\lambda_c < \infty$ or $\lambda_c = \infty$?

If μ has a density and a hazard rate that is bounded and decreasing, then $\lambda_c(\mu) < \infty$ (Fontes, Mountford, Ungaretti and Vares (2023)).

Renewal Contact Process - Some important results

Theorem 5: [Santos, Vares (to appear)]

If μ is a continuous distribution satisfying $\mu[0, b] = 1$ for some $b < \infty$, then $\lambda_c(\mu) < \infty$ (valid for any $d \geq 1$).

Theorem 6: [Santos, Vares (to appear)]

If μ is an absolutely continuous distribution on $[0, \infty)$ with density f and distribution function F such that its hazard rate $\frac{f(t)}{1-F(t)}$ is decreasing in t, then $\lambda_c(\mu) < \infty$ (valid for any $d \geq 1$).

Renewal Contact Process - Proof ideas

- Both proofs involve coupling the RCP(μ) with supercritical oriented percolation.
- We create some convenient vertical and horizontal boxes in $\mathbb{Z}^+ \times \mathbb{R}^+$. The boxes will be classified as "good" or "bad" depending if there is a path that allows the infection to cross the box in the desired direction.
- These boxes will be chosen in such a way that if we can find an infinite path constructed only using good boxes, then the RCP(μ) survives.
- For any fixed $\epsilon > 0$, we can find $\tilde{\lambda}(\epsilon) < \infty$ such that the probability of any box (vertical or horizontal) to be good is greater than $1 - \epsilon$ if we take $\lambda \geq \tilde{\lambda}(\epsilon)$.
- Events involving boxes with edges in the same vertical line are not independent. We deal with this problem in each Theorem with a different approach.

Renewal Contact Process - Proof ideas

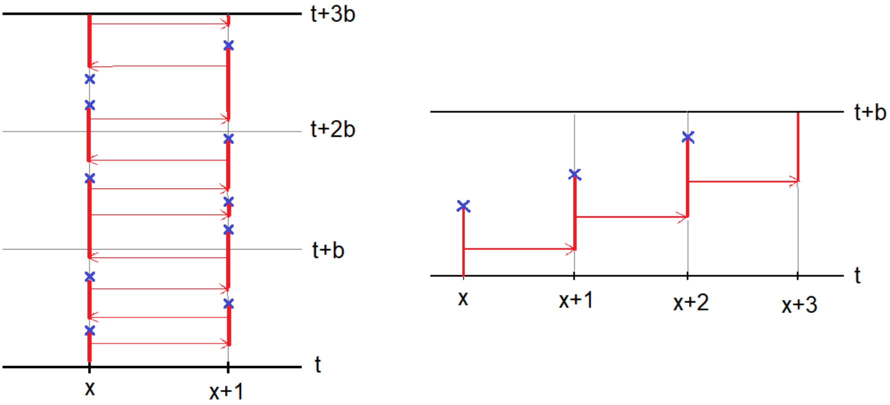

Figure: Event “The box is good”

Contact Process - References

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