Targeted cutting of random recursive trees.

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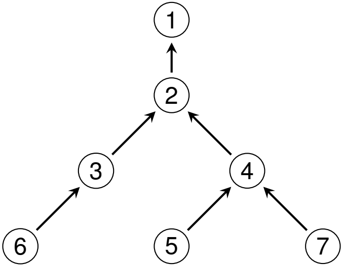
Outline

1. Random cutting (Fortuitous failures)
2. Targeted cutting (Malicious attack)
3. Results
4. Coupling
5. Work in progress

Increasing trees

- Edges directed towards root
- Labels $[n] = \{1, 2, \ldots, n\}$
- Labels increase away from root
- $deg_{T_n}(v)$, $\Delta(T_n)$



Building a random recursive tree

A random recursive tree on n vertices is a tree chosen uniformly at random on the class of increasing trees on $[n]$.

- T_1 is a tree with a single vertex

For $i > 1$ build T_i from T_{i-1} by adding

- vertex i
- arrow $i \to j$

$$\mathbf{P}\left(i \to j | T_{i-1}\right) = \frac{1}{i-1}$$

Building a RRT

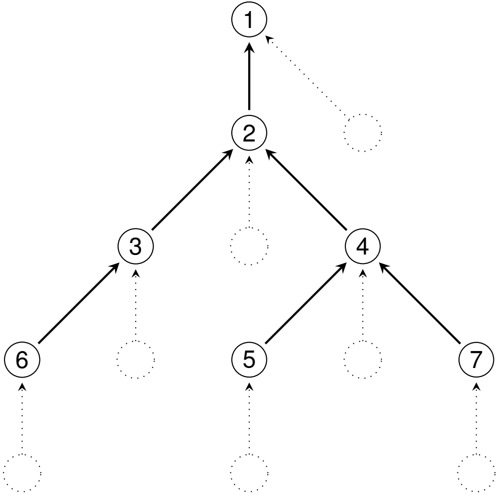


Figure: Obtaining T_8 from T_7

Characteristics

$$D := deg_{T_n}(1) = \sum_{i=2}^{n} B_i, \text{ where } B_i \text{ is a } Bernoulli \left(\frac{1}{i-1}\right)$$

$$\mathbb{E}D = H_{n-1} \sim \ln(n)$$

Goh and Schmutz [2002]. Let $\Delta(T_n)$ be the maximum degree in T_n. Then

$$\Delta(T_n) \sim \log_2(n) \approx 1.4 \ln(n)$$

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Na and Rapoport [1970]

$$Z_i(n) := \#\{1 \leq j \leq n \mid deg_{T_n}(j) = i\},$$

$$\frac{\mathbb{E}Z_i(n)}{n} \xrightarrow{n\to\infty} \frac{1}{2^{i+1}},$$

$$\mathbb{E}Z_i(n) \xrightarrow{n\to\infty} \begin{cases} \infty & \text{for} \quad i << \log_2(n) \\ \frac{1}{2} & \text{for} \quad i - \log_2(n) = \Theta(1) \\ 0 & \text{for} \quad i >> \log_2(n) \end{cases}$$

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Addario-Berry and Eslava [2017]

$$Z_{\ge i}(n) = \#\{1 \le j \le n \mid deg_{T_n}(j) \ge i\}.$$

$$\mathbb{E}Z_{\ge k+\lfloor \log_2(n)\rfloor}(n) \xrightarrow{n\to\infty} \frac{1}{2^k}$$

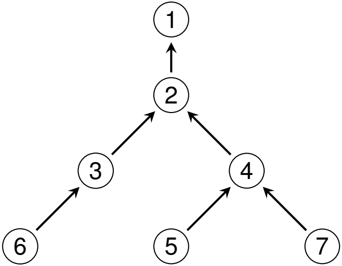
for $k \in \mathbb{N}$ and $k = k(n)$ in a suitable range.

$$\mathbb{E}Z_i(n) \xrightarrow{n\to\infty} \begin{cases} \infty & \text{for} \quad i << \log_2(n) \\ \frac{1}{2} & \text{for} \quad i - \log_2(n) = \Theta(1) \\ 0 & \text{for} \quad i >> \log_2(n) \end{cases}$$

Random cutting

Introduced by Meir and Moon [1974]

- Choose an edge uniformly at random and delete it
- Discard the subtree not containing the root
- Repeat this procedure until the root is isolated



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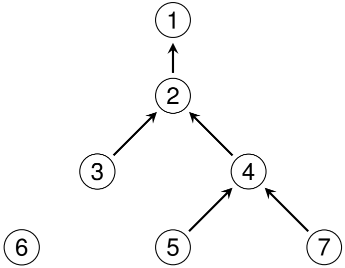

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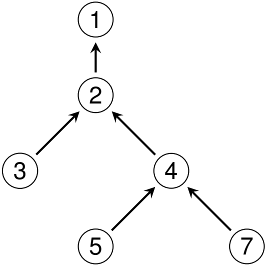

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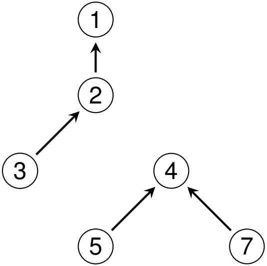

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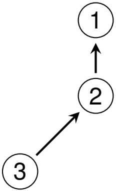

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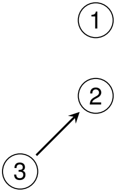

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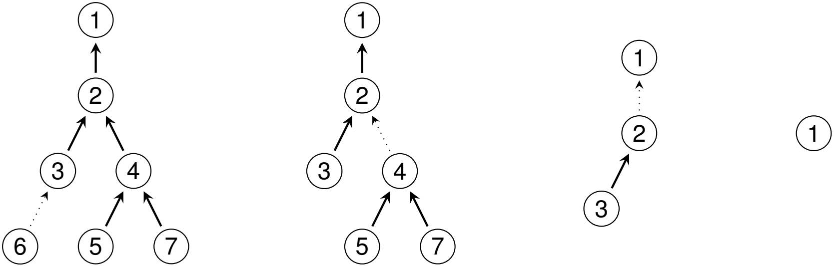


Let $J_n = \#$ cuts needed to destroy a random recursive tree with n vertices.

J_n

Key property: After each cut the remaining tree is a RRT. Used to obtain

$$\mathbb{E}J_n \sim \frac{n}{\ln n}$$

Meir and Moon [1974]

$$\mathbb{E}J_n \sim \frac{n}{\ln(n)}$$

Panholzer [2004]

$$\frac{\ln(n)}{n} J_n \xrightarrow{P} 1$$

Drmota et al. [2009]

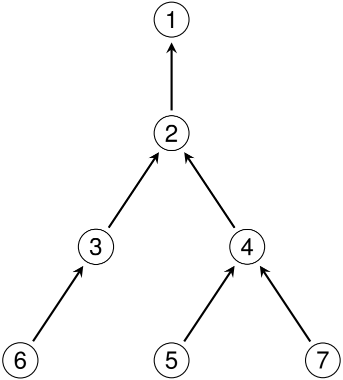
$$\frac{(\ln(n))^2}{n} J_n - \ln(n) - \ln(\ln(n))$$

converges weakly to a random variable Y with characteristic function

$$\varphi_Y(\lambda) = \exp\left\{ i\lambda \ln|\lambda| - \frac{\pi|\lambda|}{2} \right\}$$

Our proposal: Targeted cutting

- List the vertices from highest to lowest degree
- Sequentially remove the vertex in the listed order, discarding the subtrees not containing the root
- End when the root is removed



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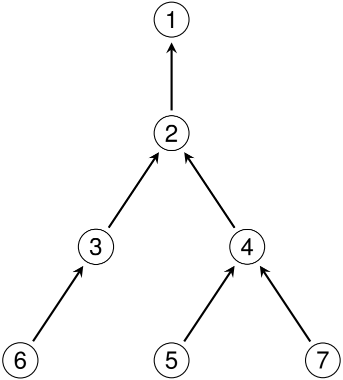
4, 2, 3, 1, 7, 6, 5



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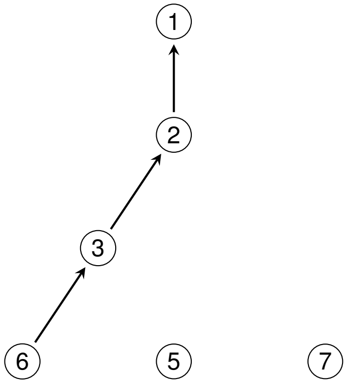

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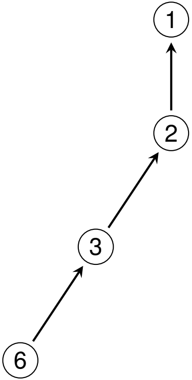

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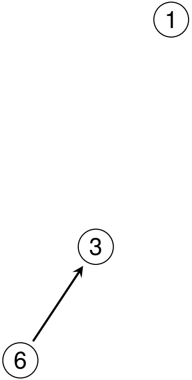

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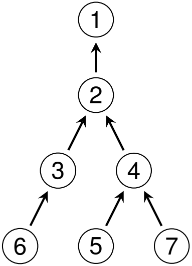

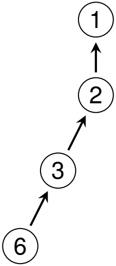

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Our proposal: Targeted cutting

We lose properties that were known in the random cutting procedure, including

- The distribution of the remaining tree
- The number of vertices deleted after each cut

Let $K_n = \#$ targeted cuts needed to delete a random recursive tree with n vertices.
Recall

- $Z_{\geq i}(n) = \#\{1 \leq j \leq n \mid deg_{T_n}(j) \geq i\}$
- $D = deg_{T_n}(1)$

Note

$$K_n \leq Z_{\geq D}(n).$$

Results

Let $\gamma := 1 - \ln(2) \approx 0.307$

Theorem (Moments)

For any positive integer k,

$$\mathbb{E}\big(\ln(Z_{\geq D})\big)^k = \big(\ln(n^\gamma)\big)^k (1 + O(1)).$$

Theorem (Convergence in probability)

$$\frac{\ln(Z_{\geq D})}{\ln(n)} \xrightarrow{P} \gamma.$$

Theorem (Growth order)

For any $\varepsilon > 0$,

$$K_n = O_p(n^{\gamma+\varepsilon}).$$

Proof strategy

Conditioned on the event

$$(1-\varepsilon)\ln(n) \leq D \leq (1+\varepsilon)\ln(n),$$

which we denote $D \in (1 \pm \varepsilon)\ln(n)$, we have that

$$Z_{\geq \lceil (1+\varepsilon)\ln(n)+1 \rceil} \leq Z_{\geq D} \leq Z_{\geq \lfloor (1-\varepsilon)\ln(n) \rfloor}.$$

Moments of the upper bound can be controlled [Addario-Berry and Eslava, 2017].
Lower bound can be approximated with a Poisson random variable [Eslava, 2020].

Coupling

The idea is to build simultaneously T_n, a RRT, and $T_n^{(\varepsilon)}$, a RRT conditioned on $D^{(\varepsilon)} \in (1 \pm \varepsilon)\ln(n)$.

- $B = (B_2, \ldots, B_n)$ with indepent entries where B_i is Bernoulli $\left(\frac{1}{i-1}\right)$.
- $B^{(\varepsilon)} = (B_2^{(\varepsilon)}, \ldots, B_n^{(\varepsilon)})$ with the law of B conditioned on $\sum_{i=2}^n B_i^{(\varepsilon)} \in (1 \pm \varepsilon)\ln(n)$.
- Y_i uniform in $\{2, 3, \ldots, i-1\}$ for $i \geq 3$.

Coupling

$$Y_3 = 2,\ Y_4 = 2,\ Y_5 = 4,\ Y_6 = 3,\ Y_7 = 4$$

$B = (1, 0, 0, 0, 0, 0)$ $\quad$ $B^{(\varepsilon)} = (1, 0, 1, 0, 0, 0)$

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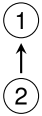

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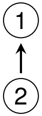

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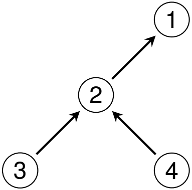

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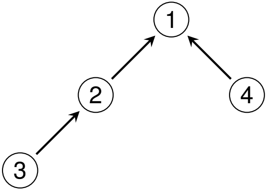

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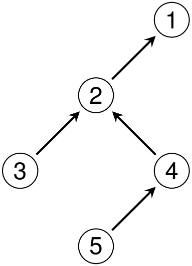

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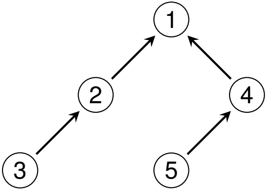

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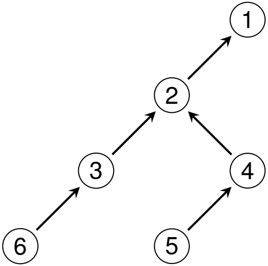

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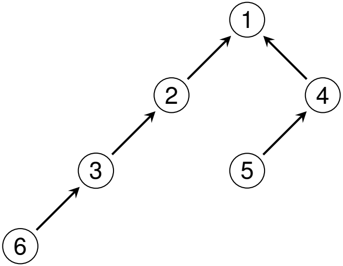

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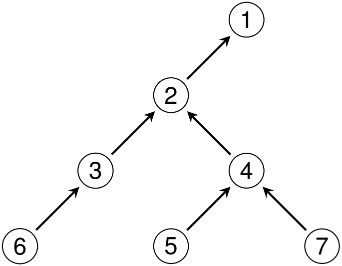

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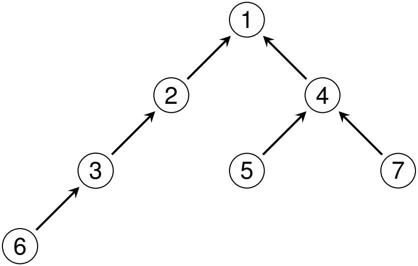

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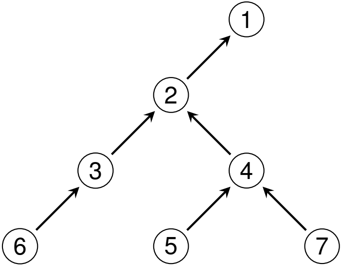
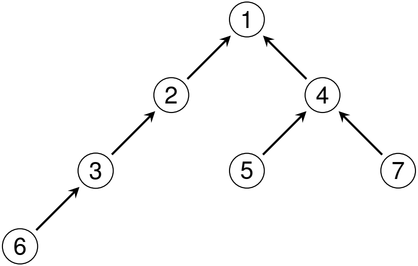


T_n

$B^{(\varepsilon)} = (1, 0, 1, 0, 0, 0)$



$T_n^{(\varepsilon)}$

Under this coupling

Let $Z^{(\varepsilon)}_{\geq i}(n) = \#\{1 \leq j \leq n \mid deg_{T_n^{(\varepsilon)}}(j) \geq i\}$, $W_d = \dfrac{1 + Z^{(\varepsilon)}_{\geq d}(n)}{1 + Z_{\geq d}(n)}$

Proposition (Similitud en razón)

Let $\varepsilon \in (0, 1/3)$, $\delta \in (0, 1)$. *There exists* $\beta_3 \in (0, \varepsilon^2/12)$, *C constant y* $n_0 = n_0(\varepsilon, \delta)$ *such that* $0 < d < (1+\varepsilon)\ln(n)$ *under the coupling*

$$\mathbb{P}\left(W_d \in (1 \pm \delta)\right) \geq 1 - Cn^{-\beta_3}.$$

for each $n > n_0$

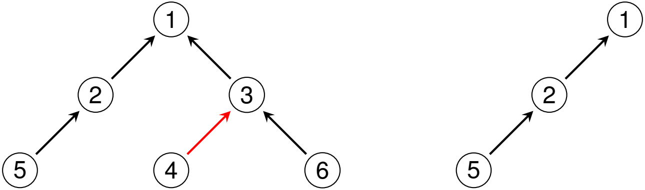
Proposition

Let $\varepsilon \in (0, 1/3)$, $k \in \mathbb{N}$. *For* $0 < d < (1+\varepsilon)\ln(n)$ *under the coupling*

$$\left|\mathbb{E}(\ln(W_d))^k\right| \leq 2C(\ln(n+1))^k n^{-\beta_3}.$$

Work in progress: Degree biased random cutting

- Choose an edge uniformly at random and delete the vertex closest to the root.
- Keep the subtree having the root.
- The process is finished when the root is deleted.



$$\frac{(\ln n)^2}{n}\mathbb{E}(Z_n) \to 1.$$

Open problems

- $\delta-$atack: Generalized cutting process. Let $\delta \in (0,1)$ and T_n a RRT.
  - List the vertices by their degree.
  - Remove a vertex uniformly from $\{v_1, \dots, v_{\lceil \delta n \rceil}\}$.
  - End the process when the root is removed.

  Case $\delta \geq 1 - \frac{1}{n}$ is the random atack, case $\delta \leq \frac{1}{n}$ is targeted cutting. Interesting case: $\delta = n^{-\alpha}$ for some $\alpha > 0$.
- Weighted random trees.

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Recapitulation

- Number of random cuts needed to destroy a RRT of size n:

$$\mathbb{E}J_n \sim \frac{n}{\ln(n)}.$$

$$\frac{\ln(n)}{n} J_n \xrightarrow{P} 1.$$

$$\frac{(\ln(n))^2}{n} J_n - \ln(n) - \ln(\ln(n)) \xrightarrow{d} Y.$$

- Number of targeted cuts needed to destroy a RRT K_n:
  For any $\varepsilon > 0$,

$$K_n = O_p\left(n^{1-\ln(2)+\varepsilon}\right).$$

- $K_n \leq Z_{\geq D}(n)$:

$$\frac{\ln(Z_{\geq D}(n))}{\ln(n)} \xrightarrow{P} 1 - \ln(2).$$

$$\mathbb{E}\bigg(\ln(Z_{\geq D}(n))\bigg)^k \sim \left(\ln\left(n^{1-\ln(2)}\right)\right)^k.$$