On a Dyson fractional Brownian motion

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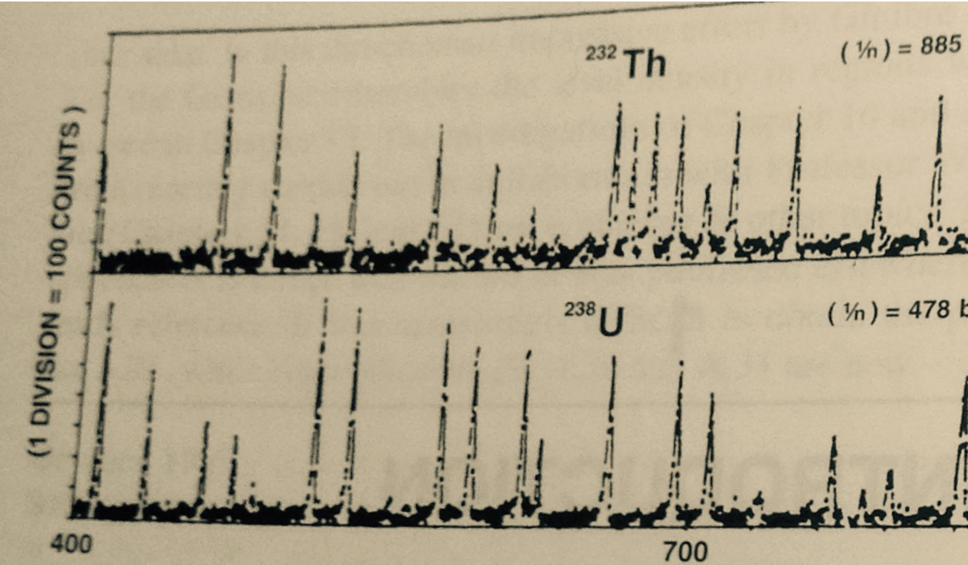
(Recalling Semicircle Law for Random Matrices)

0.A. Random matrices and nuclear physics

Slow neutron resonance cross-sections on thorium 232 & uranium 238 nuclei. Energy(eV)



0.A. Gaussian Orthogonal Ensemble (GOE)

- Ensemble: $\mathbf{Z} = (Z_n)$, Z_n is $n \times n$ matrix with random entries.
- A) GOE: $Z_n = (Z_n(j,k))$ is $n \times n$ symmetric matrix with independent Gaussian entries in the upper triangular part:

$$Z_n(j,k) = Z_n(k,j) \sim N(0,1), \quad j \neq k,$$
$$Z_n(j,j) \sim N(0,2).$$

- B) Distribution of Z_n is orthogonal invariant: $Z_n O^\top$ & Z_n have same distribution for each orthogonal matrix O.
- Characterization GOE: A and B holds.

0.A. Gaussian Orthogonal Ensemble (GOE)

- Joint density of eigenvalues of $\lambda_1 > \ldots > \lambda_n$ of Z_n:

$$f_{\lambda_1,\ldots,\lambda_n}(x_1,\ldots,x_n) = k_n \underbrace{\left[\prod_{j=1}^{n} \exp\left(-\frac{1}{4}x_j^2\right)\right]}_{\text{independence}} \underbrace{\left[\prod_{j<k} |x_j - x_k|\right]}_{\text{strong dependence}}$$

- Non-diagonal RM with density: eigenvalues are strongly dependent due to Vandermonde determinant:
  $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$

$$\Delta(x) = \det\left(\left\{x_j^{k-1}\right\}_{j,k=1}^{n}\right) = \prod_{j<k} |x_j - x_k| \,.$$

0.A. Wigner semicircle law

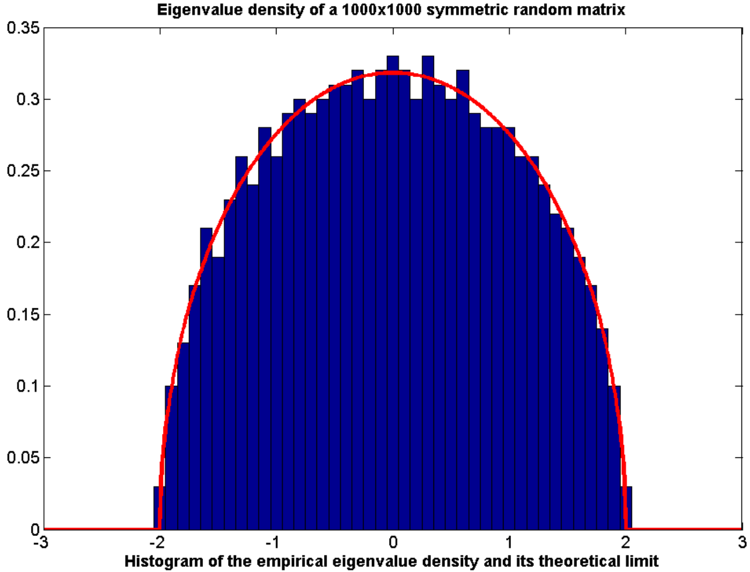
Wigner 1950s: Birth of RMT when both dimensions goes to ∞.

- A heavy nucleus is a liquid drop composed of many particles with unknown strong interactions,
- so a random matrix would be a possible model for the Hamiltonian of a heavy nucleus.
- Which random matrix should be used?
- $\lambda_1 > ... > \lambda_n$ eigenvalues of scaled GOE: $X_n = Z_n/\sqrt{n}$.
- Sample Spectral Distribution $\widehat{F}_n^{X_n}$
- Limiting Spectral Distribution (**LSD**): $\widehat{F}_n^{X_n}$ goes, as $n \to \infty$, to Semicircle distribution on $(-2, 2)$

$$w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \quad |x| \le 2.$$

- Semicircle distribution is also called free Gaussian distribution

0.A. Simulation of Wigner semicircle law



0.A. Precise statement of Wigner semicircle law

Semicircle distribution approximates the spectral distribution

Theorem: For each continuous bounded function f and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \int f(x) \mathrm{d}\widehat{F}_n^{X_n}(x) - \int f(x) w(\mathrm{d}x) \right| > \varepsilon \right) = 0$$

where $w(x)$ is the density of *semicircle distribution on* $(-2, 2)$

$$w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \quad |x| \leq 2$$

- Good predictions for moderate dimension n
- Breakthrough work by Eugene Wigner: *Ann. Math.*, 1955, 1957, 1958

0.A. Gaussian Unitary Ensemble (GUE)

Wigner law also holds

- GUE: $Z_n = (Z_n(j,k))$ is $n \times n$ Hermitian with independent Gaussian entries:

$$Z = \begin{pmatrix} Z_n(1,1) & Z_n(1,2) & \dots & Z_n(1,n) \\ \overline{Z}_n(1,2) & Z_n(2,2) & & \\ & & & \\ \overline{Z}_n(1,n) & & & Z_n(n,n) \end{pmatrix}$$

$$\mathrm{Re}\,(Z_n(j,k)) \sim \mathrm{Im}\,(Z_n(j,k)) \sim N(0, (1+\delta_{jk}))/2),$$

$$\mathrm{Re}\,(Z_n(j,k))\,, \mathrm{Im}\,(Z_n(j,k))\,, 1 \le j \le k \le n,$$

are independent random variables

- Distribution of Z_n is unitary invariant: $Z_n U^*$ & Z_n have same distribution for each unitary non-random matrix U

0.A. Universality

- Wigner semicircle law holds for **Wigner ensembles**:

$$X_n(k,j) = X_n(j,k) = \frac{1}{\sqrt{n}} \begin{cases} Z_{j,k}, & \text{if } j < k \\ Y_j, & \text{if } j = k \end{cases}$$

$\{Z_{j,k}\}_{j \leq k}, \{Y_j\}_{j \geq 1}$ independent sequences of i.i.d. r.v. with

$$\mathbb{E} Z_{1,2} = \mathbb{E} Y_1 = 0, \mathbb{E} Z_{1,2}^2 = 1, \mathbb{E} Y_1^2 < \infty$$

- Whatever values the random entries take, the **LSD** (Semicircle) has bounded support
- Joint density of eigenvalues of a Wigner matrix is not easy

0. The Marchenko-Pastur Law

B. For Sample Covariance Matrix

Both, data dimension and sample size large

0.B. Marchenko-Pastur law

Marchenko-Pastur (1967), *Mat. Sb.*

- $H = H_{p \times n} = (Z_{j,k} : j = 1, .., p, k = 1, ..., n)$ i.i.d. r.v.

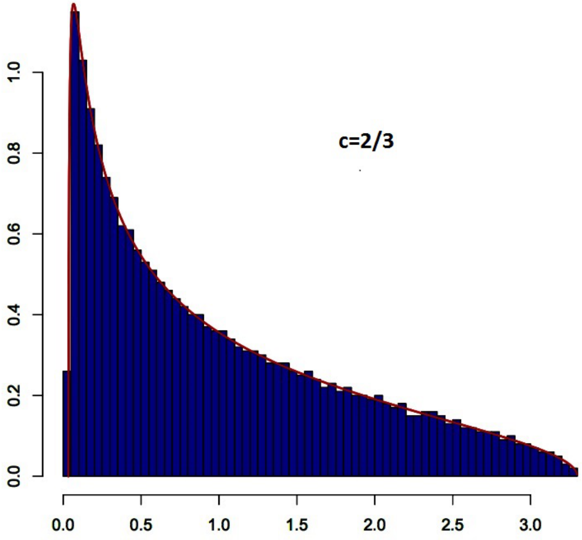
$$\mathbb{E}(Z_{1,1}) = 0,\ \mathbb{E}(|Z_{1,1}|^2) = 1,\ \mathbb{E}(|Z_{1,1}|^4) < \infty$$

- Sample covariance matrix $S_n = \frac{1}{n} HH^*$, ESD $\widehat{F}_p^{S_n} = \widehat{F}_p^{\frac{1}{n}HH^*}$
- If $p/n \to c > 0$, $\widehat{F}_p^{S_n}$ goes to **MP distribution**:

$$\mu_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \textit{if} \quad 0 < c < 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \textit{if} \quad c \geq 1 \end{cases}$$

$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\mathbf{1}_{[a,b]}(x)$$
$$a = (1-\sqrt{c})^2, \quad b = (1+\sqrt{c})^2$$

0.B. Simulation Marchenko-Pastur law



0. Random matrices have been used in many fields

- Statistics
- Physics
- Number theory
- Biology
- Finances
- Engineering
- Computer vision
- Machine learning
- ......
- Prominently...together with *free probability* in *wireless communications:*

Time-varying random matrices: why?

Couillet & Debbah (2011), *Random Matrix Methods for Wireless Communications.* Chapter 19, Perspectives:

- Performance analysis of a typical network with users in motion according to some stochastic behavior, is not accessible to this date in the restrictive framework of random matrix theory.

- It is to be believed that random matrix theory for wireless communications may move on a more or less long-term basis towards random matrix process theory for wireless communications. Nonetheless, these random matrix processes are nothing new and have been the interest of several generations of mathematicians.

- The work was initiated by Dyson in 1962

Part I: From Matrix Bm to Free Bm

(Time-varying random matrix models for the Free Brownian motion)

I. Hermitian Brownian motion ensemble

- $\mathbf{B}(t) = (B_n(t))_{n\geq 1},\ t \geq 0.$
- $B_n(t)$ is $n \times n$ Hermitian Brownian motion:

$$B_n(t) = (b_{ij}(t)), t \geq 0,$$

$$\mathrm{Re}(b_{ij}(t)) \sim \mathrm{Im}(b_{ij}(t)) \sim N(0, t(1+\delta_{ij}))/2,$$

where $\mathrm{Re}(b_{ij}(t)), \mathrm{Im}(b_{ij}(t)), 1 \leq i \leq j \leq n$ are independent one-dimensional Brownian motions

- $(\lambda_1(t), ..., \lambda_n(t))_{t\geq 0}$ process of eigenvalues *of* $\{B_n(t)\}_{t\geq 0}$

$$\lambda_1(t) \geq \lambda_2(t) \geq ... \geq \lambda_n(t)$$

I. Dyson-Brownian motion

Time dynamics of the eigenvalues, dimension n fixed

Dyson (1962):

a) If eigenvalues start at different positions, they never collide

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > ... > \lambda_n(t) \quad \forall t > 0\right) = 1$$

b) They satisfy the Stochastic Differential Equation (SDE)

$$\lambda_i(t) = \lambda_i(0) + W_i(t) + \sum_{j \neq i} \int_0^t \frac{\mathrm{d}s}{\lambda_j(s) - \lambda_i(s)}, i = 1, ..., n$$

$\forall t > 0$, where $W_1, ..., W_n$ are 1-dimensional independent Bms

- Brownian part + repulsion force (at any time t)
- Proof uses *classical stochastic calculus* and *martingale* techniques (Anderson, Guionnet, Zeitouni, 2010, Tao, 2012)

I. Key ideas for the proof

Where does the noncoliding force come from?

1. Eigenvalues are smooth functions of entries:

$$\lambda_i(t) = F(B_n(t)) = F((b_{jk}(t)))$$

2. *Itô formula* from *classical stochastic calculus*: $b = (b_t)_{t \geq 0}$ 1-dimensional Bm

$$F(b_t) = \int_0^t F_x(b_s)\mathrm{d}b_s + \frac{1}{2}\int_0^t F_{xx}(b_s)\mathrm{d}s$$

3. *Hadamard second variational formula*: For a matrix $A = A(x)$ depending smoothly on x, and $Au_i = \lambda_i u_i$, $u_i^* u_i = 1$

$$\frac{d^2}{dx^2}\lambda_i = u_i^* A_{xx} u_i + 2\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \left| u_j^* A_x u_j \right|$$

4. Existence of independent Bms $W_1, \ldots, W_n$ from Lévy's characterization of Bm

I. Time-varying Wigner theorem and law of Free Bm

- Consider the Dyson spectral measure-valued processes

$$\mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\{\lambda_j(t)/\sqrt{n}\}}, \quad t \geq 0, n \geq 1$$

- Notation: For f μ-integrable function $\langle \mu, f \rangle = \int f(x)\mu(\mathrm{d}x)$
- Uniform Wigner theorem

$$\mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq t\leq T} \left| \left\langle \mu_t^{(n)}, f \right\rangle - \langle \mathrm{w}_t, f \rangle \right| = 0, \forall f \in C_b(\mathbb{R}) \right) = 1$$

- The family of probability measures $\{\mathrm{w}_t\}_{t\geq 0}$ is the Law of the Free Brownian motion,

$$\mathrm{w}_t(\mathrm{d}x) = \frac{1}{2\pi t}\sqrt{4t - x^2}1_{[-2\sqrt{t},2\sqrt{t}]}(x)\mathrm{d}x$$

I. Smooth vs non smooth SDE

Interacting systems

- SDE with both **smooth drift & diffusion coefficients** β and α are of the form

$$X_{n,i}(t) = X_{n,i}(0) + \frac{1}{\sqrt{n}} \sum_{j \neq i} \int_0^t \beta(X_{n,j}(s), X_{n,i}(s)) \mathrm{d}W_i^{(n)}(t)$$
$$+ \frac{1}{n} \sum_{j \neq i} \int_0^t \alpha(X_{n,j}(s), X_{n,i}(s)) \mathrm{d}s.$$

- While Dyson-Brownian motion has **very non smooth drift**

$$X_{n,i}(t) = X_{n,i}(0) + \frac{1}{\sqrt{n}} W_i^{(n)}(t) + \frac{1}{n} \sum_{j \neq i} \int_0^t \frac{1}{X_{n,i}(s) - X_{n,j}(s)} \mathrm{d}s.$$

- Empirical measure valued processes

$$\mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_{n,j}(t)}, \quad t \geq 0, n \geq 1$$

I. Smooth vs non smooth SDE

1. **For Interacting SDE with both smooth drift & diffusion coefficients**:

- McKean (Lect. Series Differ. Equat. 1967): $\left\{\mu^{(n)}\right\}_{t\geq 0}$ converges weakly in probability to $\{\mu_t\}_{t\geq 0}$, which is the law of a Itô stochastic differential equation.

2. **Interacting SDE with non-colliding forces arise from eigenvalue processes of matricial processes:**

- [Bru (1989), Rogers & Shi (1993), Konig & O´Connell (2001), Cabanal-Duvillard & Guionnet (2001), Katori & Tanemura (2004)].
- The family of probabilities $\{\mathrm{w}_t, t \geq 0\}$ is not the law of a SDE equation, but the law of a noncommutative process: Free Brownian motion

I. Fluctuations limit: Another difference

1. Interacting SDE with **smooth drift & diffusion coefficients:**

- Limits of fluctuations (CLT): $S_t^{(n)} = \sqrt{n}(\mu_t^{(n)} - \mu_t)$.
- Hitsuda and Mitoma (JMA, 1986): $S_t^{(n)}$ converges to **Gaussian process in nuclear space** (Kallianpur & Pérez-Abreu (AMO, 1988), Kallianpur & Xiong (LNS, 1995))
- Gaussian fluctuations for interacting particle systems with a class of singular kernels, Wang, Zhao & Zhu (2021)

2. Interacting SDE with **non-colliding eigenvalues:**

- Due to noncoliding forces, need to consider fluctuations

$$Y_t^{(n)} = n\left(\mu_t^{(n)} - \mathrm{w}_t\right)$$

- Israelson (SPA, 2001), Bender (SPA, 2008): $Y_t^{(n)}$ converges to **Gaussian process in nuclear space**
- Unterbergerg (SPA, 2018): Global fluctuations for 1D log-gas dynamics (generalized Dyson-Brownian motion)

What is free Brownian motion?

What is the law of free Brownian motion?

I. Notation

- Let $\mathcal{P}(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$.
- Let $C(\mathbb{R}_+, \mathcal{P}(\mathbb{R}))$ be the spaces of continuous functions from $\mathbb{R}_+ \to \mathcal{P}(\mathbb{R})$, with the topology of uniform convergence on compact intervals of $\mathbb{R}_+$
- For $\mu \in \mathcal{P}(\mathbb{R})$ and a function $f : \mathbb{R} \to \mathbb{R}$ that is μ-integrable we write

$$\langle \mu, f \rangle = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x)$$

- Cauchy transform of $\mu \in \mathcal{P}(\mathbb{R}) : G_\mu(z) : \mathbb{C}^+ \to \mathbb{C}^-$

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x}\mu(\mathrm{d}x), \quad z \in \mathbb{C}^+$$

I. Law of free Bm as limiting measure valued process

Theorem (Cabanal-Duvillard & Guionnet (2001))

If $\mu_0^{(n)} \to \delta_0$, *the family* $\left(\mu_t^{(n)}\right)_{t\geq 0}$ *of measure valued-processes converges weakly in* $C\left(\mathbb{R}_+, \mathcal{P}(\mathbb{R})\right)$ *to a unique continuous probability-measure valued function such that* $\forall\ f \in C_b^2(\mathbb{R})$

$$\langle \mu_t, f\rangle = f(0) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \frac{f'(x)-f'(y)}{x-y}\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y).$$

Moreover, $\mu_t = \mathrm{w}_t,\ t \geq 0$

Well known fact: The family of laws $(\mathrm{w}_t)_{t\geq 0}$ is characterized by its Cauchy transforms $(G_t)_{t\geq 0}$, being the unique solution of

$$\frac{\partial G_t(z)}{\partial t} = G_t(z)\frac{\partial G_t(z)}{\partial z}, \quad t > 0$$
$$G_0(z) = -\frac{1}{z}, \quad z \in \mathbb{C}^+,$$

$G_t(z) \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$ & $\lim_{\eta\to\infty} \eta\,|G_t(i\eta)| < \infty\ \forall\ t > 0$

Part II:

Free Brownian motion

and

Non-commutatitve fractional Brownian motion

II. Non-commutatitve probability spaces

A *non-commutatitve probability space* $(\mathcal{A}, \varphi)$ is a unital algebra $\mathcal{A}$ over $\mathbb{C}$ with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(1_{\mathcal{A}}) = 1$. Elements of $\mathcal{A}$ are called *non-commutatitve random variables*

- **Examples**

1. $\mathcal{A} = \mathbb{M}_d(\mathbb{C})$ $d \times d$ matrices with complex entries

$$\varphi(\cdot) = \mathrm{tr}_d(\cdot) = \frac{1}{d}\mathrm{tr}(\cdot)$$

2. $\mathcal{A} = L_\infty(\Omega, \mathcal{F}, \mathbb{P})$,

$$\varphi(\cdot) = \mathbb{E}(\cdot)$$

3. $\mathcal{A} = \mathbb{M}_d(L_\infty(\Omega, \mathcal{F}, \mathbb{P}))$,

$$\varphi(\cdot) = \mathbb{E}\mathrm{tr}_d(\cdot)$$

4. $\mathcal{A} = L(\mathcal{H})$ algebra of linear operators on a Hilbert space, $u \in H$, $\|u\| = 1$

$$\varphi(\cdot) = \langle \cdot u, u \rangle$$

- We should think of φ as playing the role of the expectation in classical probability theory
- We talk about the moments of a, referring to the values of $\varphi\left(a^k\right), \quad k \geq 0$
- More generally, for a tuple $a_1, \ldots, a_n \in \mathcal{A}$, the values

$$\varphi(a_{i_1}^{m_1} ... a_{i_k}^{m_k})$$

  for $k \geq 0$, $1 \leq i_1, \ldots, i_k \leq n$, $m_1, \ldots m_k \geq 0$, are known as the *joint moments* of $a_1, \ldots, a_n$
- When an algebraic distribution is given by an analytic distribution?

II. Non-commutatitve probability spaces

Generality needed to deal with free probability

Remember classical case: A real random variable R has distribution μ on $\mathbb{R}$ iff

$$\mathbb{E}f(R) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in B_b(\mathbb{R})$$

Non-commutatitve case needs:

(i) **Given a p.m.** μ on $\mathbb{R}$ with **bounded support**, there exist a C^*-probability space $(\mathcal{A}, \varphi)$ and a self-adjoint $\mathbf{a} \in \mathcal{A}$ with

$$\varphi(f(\mathbf{a})) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in C_b(\mathbb{R})$$

(ii) **Given a p.m.** μ on $\mathbb{R}$, there exists a W^*-probability space $(\mathcal{A}, \varphi)$ and self-adjoint operator $\mathbf{a}$ on a Hilbert space H such that

$$f(\mathbf{a}) \in \mathcal{A} \quad \forall f \in B_b(\mathbb{R}), \tag{1}$$

$$\varphi(f(\mathbf{a})) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in B_b(\mathbb{R})$$

If (1) holds, it is said that $\mathbf{a}$ *is affiliated with* $\mathcal{A}$

II. Free Random Variables

Definition

A family of subalgebras $\{\mathcal{A}_i\}_{i \in I} \subset \mathcal{A}$ in a non-commutatitve probability space is *free* **(freely independent)** if

$$\varphi(\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n) = 0$$

whenever $\varphi(\mathbf{a}_j) = 0$, $\mathbf{a}_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$

Definition

If $\mathbf{a}_1, \mathbf{a}_2$ are freely independent, with distributions μ_{a_1} and μ_{a_2}, the distribution of $\mathbf{a}_1 + \mathbf{a}_2$ is the free convolution $\mu_{a_1} \boxplus \mu_{a_2}$

- **Recall:** If $\mathbf{a}_1, \mathbf{a}_2$ are classical independent, the distribution of $\mathbf{a}_1 + \mathbf{a}_2$ is the classical convolution $\mu_{a_1} * \mu_{a_2}$
- If $\varphi = \mathbb{E}$ and there is commutativity

$$\mathbb{E}(\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n) = \mathbb{E}(\mathbf{a}_1) \cdots \mathbb{E}(\mathbf{a}_n) = 0$$

II. Free independence allows to compute joint moments

Example

Computation of $\varphi(\mathbf{abab})$ when $\mathbf{a}$ & $\mathbf{b}$ are freely independent:
Suppose $\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_2, \mathbf{a}_4\}$ are freely independent. Since

$$\varphi(\mathbf{a}_i - \varphi(\mathbf{a}_i)1_{\mathcal{A}}) = 0,$$

$$\varphi(\mathbf{a}_1 - \varphi(\mathbf{a}_1)1_{\mathcal{A}})\varphi(\mathbf{a}_2 - \varphi(\mathbf{a}_2)1_{\mathcal{A}})\varphi(\mathbf{a}_3 - \varphi(\mathbf{a}_3)1_{\mathcal{A}})\varphi(\mathbf{a}_4 - \varphi(\mathbf{a}_4)1_{\mathcal{A}}) = 0$$

Computations yield

$$\varphi(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4) = \varphi(\mathbf{a}_1\mathbf{a}_3)\varphi(\mathbf{a}_2)\varphi(\mathbf{a}_4) + \varphi(\mathbf{a}_1)\varphi(\mathbf{a}_3)\varphi(\mathbf{a}_2\mathbf{a}_4) - \varphi(\mathbf{a}_1)\varphi(\mathbf{a}_2)\varphi(\mathbf{a}_3)\varphi(\mathbf{a}_4)$$

In particular if $\mathbf{a}_1 = \mathbf{a}_3 = \mathbf{a}$ and $\mathbf{a}_2 = \mathbf{a}_4 = \mathbf{b}$

$$\varphi(\mathbf{abab}) = \varphi(\mathbf{a})^2\varphi(\mathbf{b}^2) + \varphi(\mathbf{a}^2)\varphi(\mathbf{b})^2 - \varphi(\mathbf{a})^2\varphi(\mathbf{b})^2 \neq \varphi(\mathbf{a}^2)\varphi(\mathbf{b}^2)$$

II. Application: Free Central Limit Theorem

Theorem

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots$ *be a sequence of independent free random variables with the same distribution with all moments. Assume that* $\varphi(\mathbf{a}_1) = 0$ *and* $\varphi(\mathbf{a}_1^2) = t$*. Then the distribution of*

$$\mathbf{Z}_m = \frac{1}{\sqrt{m}}(\mathbf{a}_1 + \ldots + \mathbf{a}_m)$$

converges, as $m \to \infty$*, to the semicircle distribution*

$$\mathrm{w}_t(x) = \frac{1}{2\pi}\sqrt{4t - x^2}, \quad |x| \leq 2\sqrt{t}$$

with moments $m_{2k+1} = 0$ *and* $m_{2k} = t^{2k}\binom{2k}{k}/(k+1)$.

- **Semicircle or Wigner distribution plays the role of classical Gaussian in free probability**
- **Marchenko-Pasture distribution (free Poisson) plays the role of Poisson distribution**

II. RMT and Free Probability: Why useful?

From the Blog of Terence Tao (Free Probability):

- The significance of free probability to random matrix theory lies in the fundamental observation that *random matrices which are independent in the classical sense, also tend to be independent in the free probabilistic sense*, in the large limit.
- Because of this, many tedious computations in random matrix theory, particularly those of an algebraic or enumerative combinatorial nature, can be done more quickly and systematically by using the *framework of free probability.*
- **Voiculescu (1991)**, Limit Laws for random matrices and free products. *Invent. Math.*
- Books on random matrices and wireless communications include free probability.

II. RMT and Free Probability: Why useful?

- Knowing eigenvalues of $n \times n$ random matrices X_n & Y_n, what are the eigenvalues of $X_n + Y_n$? $X_n Y_n$?
- In general if X_n and Y_n do not commute,

$$\lambda(X_n + Y_n) \neq \lambda(X_n) + \lambda(Y_n)$$

- However, if X_n & Y_n are *asymptotically free*, LSD of $X_n + Y_n$ can be computed as (*free convolution)*

$$LSD(X_n + Y_n) = LSD(X_n) \boxplus LSD(Y_n)$$

- The problem is similar to the computation to the distribution of the sum of two independent random variables: product of characteristic functions or moment generating functions (classical convolution)

II. Free Brownian motion

A non-commutatitve process

A Free Brownian motion is a family $S = \{S_t\}_{t \geq 0}$ of self-adjoint random variables in a non-commutatitve probability space $(\mathcal{A}, \varphi)$ such that:

1. $S_0 = 0$
2. For $t_2 \geq t_1 \geq 0$, $S_{t_2} - S_{t_1}$ has law $\mathrm{w}_{t_2 - t_1}$
3. For all $n \geq 1$ and $t_n > \cdots > t_1 > 0$, the increments $S_{t_n} - S_{t_{n-1}}, \ldots, S_{t_2} - S_{t_1}, S_{t_1}$ are freely independent with respect to φ

- For every $t \geq 0$, S_t has semicircle law w_t of zero mean and variance $t > 0$
- One has Stochastic calculus for the free Brownian motion (Anshelevich, JFA, 2002; Biane, FICAMS, 1997; Biane & Speicher, PTRF, 1998); Kemp, Nourdin, Peccati, Speicher, AP, 2012)

II. Semicircular process

- Free Brownian motion is an example of a Semicircular process $X = \{X_t\}_{t\geq 0} \subset \mathcal{A}$, self-adjoint random variables: For every $k \geq 1$, $t_1, ..., t_k \in [0, \infty)$ and $\theta_1, .., \theta_k \in \mathbb{R}$, the non-commutatitve random variable $\theta_1 X_{t_1} + \cdots + \theta_k X_{t_k}$ has Semicircle law

$$\mathrm{w}_{m,\sigma^2}(\mathrm{d}x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-m)^2}1_{[m-2\sigma, m+2\sigma]}(x)\mathrm{d}x$$

for some $m \in \mathbb{R}$, $\sigma^2 > 0$

- The law of a centered semicircular processes ($\varphi(X_t) = 0$ for every $t > 0$) is uniquely determined by its covariance function

$$\Gamma(s,t) = \varphi(X_t X_s)$$

- Centered semicircular $X = \{X_t\}_{t\geq 0}$ has stationary increments

$$\Gamma(s,t) = \Gamma(|t-s|) = \varphi(X_{|t-s|})$$

II. Non-commutatitve Fractional Brownian Motion

Nourdin and Taqqu (JTP, 2104)

- Let $H \in (0,1)$. A noncommutative fractional Brownian motion (ncfBm) of Hurst parameter H is a centered semicircular process $S^H = \left\{S_t^H\right\}_{t\geq 0}$ in a non-commutatitve probability space $(\mathcal{A}, \varphi)$ with covariance function

$$\varphi(S_t^H S_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

- For each $t > 0$, S_t^H has the semicircle law w_t^H on $(-2t^H, 2t^H)$

$$\mathrm{w}_t^H(\mathrm{d}x) = \frac{1}{2\pi t^{2H}}\sqrt{4t^{2H} - x^2}\mathrm{d}x, \quad |x| \leq 2t^H$$

- ncfBm has stationary increments: For every $t, s > 0$

$$\varphi(\left(S_t^H - S_s^H\right)^2) = |t-s|^{2H}$$

Part III: Extensions to other matrix and non-commutatitve processes

III.A. From matrix fractional Bm to non-commutatitve fractional Bm

(Time-varying random matrix models for the non-commutatitve fractional Bm)

III.A. One-dimensional fractional Brownian motion

A one-dimensional fractional Brownian motion $b^H = \left\{ b^H(t) \right\}_{t \geq 0}$ is a zero-mean classical Gaussian process with covariance

$$\mathbb{E}(b^H(t) b^H(s)) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right)$$

- Stationary increments: For $s, t > 0$

$$\mathbb{E} \left| b^H(t) - b^H(s) \right|^2 = |t - s|^{2H}$$

- Self-similarity: $(a^{-H} b^H(at))_{t \geq 0} \overset{\text{law}}{=} (b^H(t))_{t \geq 0}$
- $H = 1/2$ is 1-dimensional Bm (independent increments)
- Itô stochastic calculus cannot be used for $H \neq 1/2$
- Need *classical* fractional stochastic calculus: Skorohod, Young

III.A. Matrix fractional Brownian motion

Consider $n(n+1)/2$ independent 1-dimensional fractional Brownian motions with $H \in (1/2, 1)$

$$\{\{b_{i,j}^H(t), t \geq 0\}, 1 \leq i, j \leq n\}$$

- $n \times n$ symmetric matrix fractional Brownian motion:

$$\mathbf{B}_n^H(t) = (B_{ij}^H(t))_{i,j=1}^n$$
$$B_{ij}^H(t) = b_{i,j}^H \text{ if } i < j$$
$$B_{ii}^H(t) = \sqrt{2} b_{i,i}^H(t)$$

- For $0 < t_1 < \cdots < t_p$, the increments $(\mathbf{B}_n^H(t_k - t_{k-1}))_{n \geq 1}$, $k = 1, \ldots, p$ are not independent
- Let $\lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_n(t)$ be the eigenvalues of $\mathbf{B}_n^H(t)$

III.A. Matrix fractional Brownian motion

Nualart and Pérez-Abreu (SPA, 2014): no collision

$H > 1/2$

1. If $\lambda_1(0) > \lambda_2(0) > ... > \lambda_n(0))$ the eigenvalues never collide:

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > ... > \lambda_n(t) \quad \forall t > 0\right) = 1 \qquad (*)$$

2. For any $t > 0$ and $i = 1, ..., n$

$$\lambda_i(t) = \lambda_i(0) + Y_i(t) + 2H\sum_{j \neq i} \int_0^t \frac{1}{\lambda_i(s) - \lambda_j(s)} \mathrm{d}s$$

$$Y_i(t) = \sum_{k \leq h} \int_0^t \frac{\partial \lambda_i(s)}{\partial b^H_{kh}(s)} \delta b^H_{kh}(s) \qquad (**)$$

- Stochastic integral in (**) is in the sense of Skorohod. Classical Itô stochastic calculus cannot be used for $H \neq 1/2$
- Proof of (*) uses the Young stochastic integral
- $Y_i(t)$ is not a fractional Brownian motion, but it is a self-similar process: $\forall\ a > 0$, $(a^{-H} Y_i(at))_{t \geq 0} \overset{\text{law}}{=} (Y_i(t))_{t \geq 0}$

III.A. Matrix fractional Brownian motion

Jaramillo and Nualart (RMTA, 2021) and others: colliding case!!

Case $H < 1/2$

- For $T > 0$

$$\mathbb{P}\left(\lambda_i(t) = \lambda_j(t) \text{ for some } t \in (0, T), \text{ and } 1 \leq i, j \leq n\right) = 1$$

- Tools: Hitting probabilities for Gaussian processes and capacity of Hausdorff dimension for measurable sets

General framework and some results of Jaramillo and Nualart:

- Matrix-valued Gaussian processes with a covariance R
- fBm real case: eigenvalues do not collide if $H > 1/2$ and collide if $H < 1/2$. Case $H = 1/2$ is Mckean (1969) method
- fBm complex case: eigenvalues do not collide if $H > 1/3$ and collide if $H < 1/3$
- Case $H = 1/3$, no collision: Lee, Song, Xiao and Yuan, 2023
- Collision of k eigenvalues: Song, Xiao and Yuan (2021)

III.A. Time-varying Wigner theorem

Pardo, Pérez G, Pérez-Abreu (JTP, 2016)

Let $H > 1/2$. Consider the empirical spectral measure-valued processes of the re-scaled matrix fractional Bm $\mathbf{B}_n^H(t)/\sqrt{n}$

$$\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{\lambda_j(t)/\sqrt{n}\}}, \;\; t \geq 0, n \geq 1$$

1. Fix $T > 0$. For all continuous bounded function f and $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T}\left|\int f(x)\mathrm{d}\mu_t^{(n)}(x) - \int f(x)\mathrm{w}_t^H(x)\mathrm{d}x\right| > \varepsilon\right) = 0$$

   where w_t^H is the semicircle distribution on $(-2t^H, 2t^H)$

2. The family of measure-valued processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ converges to $(\mathrm{w}_t^H)_{t\geq 0}$, the law of a non-commutatitve fractional Bm of Hurst parameter $H \in (1/2, 1)$

3. Proof needs non collinding eigenvalues

III.A. Precise statement

Pardo, Pérez G, Pérez-Abreu (JTP, 2016): Functional Wigner

1. The family of measure-valued empirical spectral processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ converges weakly in $C(\mathbb{R}_+, \mathcal{P}(\mathbb{R}))$ to the unique continuous probability-measure valued function $(\mu_t)_{t\geq 0}$ satisfying, for each $t \geq 0$, $f \in C_b^2(\mathbb{R})$,

$$\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + H \int_0^t ds \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} s^{2H-1} \mu_s(dx)\mu_s(dy)$$

   Moreover $\mu_t = \mathrm{w}_t^H$

2. The Cauchy transform $G_t(z) = \int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x}$ of μ_t is the unique solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} G_t(z) = Ht^{2H-1} G_t(z) \frac{\partial}{\partial z} G_t(z), & t > 0, \\ G_0(z) = \int_{\mathbb{R}} \frac{\mu_0(dx)}{z-x}, & z \in \mathbb{C}^+ \end{cases}$$

Extensions: Matrices built from independent 1-dimensional SDE driven by fBm, $H > 1/2$ (Song, Yao, Yuan, 2020).

III.B. Other Gaussian matrix processes

Jaramillo, Pardo, Pérez-G (JTP, 2019): Functional Wigner

Symmetric matrix built from independent Gaussian processes with regular covariance R

Includes fractional Brownian motion with index $0 < H < 1/2$

Proof does not need non-colliding eigenvalues and uses extended Skorohod integral

1. $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ converges weakly in $C(\mathbb{R}_+, \mathcal{P}(\mathbb{R}))$ to $(\mu_t)_{t\geq 0}$ satisfying, $t \geq 0$, $f \in C_b^2(\mathbb{R})$,

$$\langle \mu_t, f\rangle = \langle \mu_0, f\rangle + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} R'(s,s)\mu_s(dx)\mu_s(dy)$$

2. The Cauchy transform G_t of μ_t satisfies $G_t(z) = F_{R(t,t)}(z)$, $t > 0$, $z \in \mathbb{C}^+$, with F the solution of

$$\frac{d}{d\tau}F_t(z) = F_t(z)\frac{d}{dz}F_t(z),$$

$F_0(z)$ the Cauchy transform of $\mu_0, t > 0$, $z \in \mathbb{C}^+$

III.B. Other Gaussian matrix processes

Diaz, Jaramillo, Pardo (AIHPB, 2022): Gaussian fluctuations

Symmetric matrix from independent Gaussian processes with regular covariance R

$$Z_\Phi{}^{(n)}(t) = \left((Z_{\phi_1}^{(n)}(t), \dots, Z_{\phi_r}^{(n)}(t)); t \geqslant 0 \right)$$

$\Phi = (\phi_1, \dots, \phi_r)$, $r \geqslant 1, \phi_i$ in a large class of test functions

$$Z_\phi^{(n)}(t) = n \left[\int_{\mathbb{R}} \phi(t) \mu_t^{(n)}(dx) - \mathbb{E} \left(\int_{\mathbb{R}} \phi(t) \mu_t^{(n)}(dx) \right) \right]$$

1. There exists a continuous $\mathbb{R}^r$-valued centered Gaussian process $\Lambda_\Phi(t) = \left((\Lambda_{\phi_1}(t), \dots, \Lambda_{\phi_r}(t)); t \geqslant 0 \right)$ such that $Z_\phi^{(n)}(t)$ converges stably to $\Lambda_\Phi(t)$, in the topology of uniform convergence over compact sets
2. An explicit expression for covariance of $\Lambda_\Phi(t)$ is found
3. An upper bound for the total variation distance between the laws of $Z_\Phi{}^{(n)}(t)$ and $\Lambda_\Phi(t)$ is established

III.B. Other Gaussian matrix processes

Diaz, Jaramillo, Pardo (AIHPB, 2022): Gaussian fluctuations

Proof uses

- CLT in the Wiener chaos (related to Fourth Moment Problem of Nualart & Peccati (Ann Probab, 2005))

III.B. Other Gaussian matrix processes

Diaz, Jaramillo, Pardo (AIHPB, 2022): Gaussian fluctuations

Proof uses

- CLT in the Wiener chaos (related to Fourth Moment Problem of Nualart & Peccati (Ann Probab, 2005))
- Free independence and multiple Wigner integrals

III.B. Other Gaussian matrix processes

Diaz, Jaramillo, Pardo (AIHPB, 2022): Gaussian fluctuations

Proof uses

- CLT in the Wiener chaos (related to Fourth Moment Problem of Nualart & Peccati (Ann Probab, 2005))
- Free independence and multiple Wigner integrals
- Weak convergence results

III.B. Other Gaussian matrix processes

Diaz, Jaramillo, Pardo (AIHPB, 2022): Gaussian fluctuations

Proof uses

- CLT in the Wiener chaos (related to Fourth Moment Problem of Nualart & Peccati (Ann Probab, 2005))
- Free independence and multiple Wigner integrals
- Weak convergence results
- Stein methods

III.B. Other Gaussian matrix processes

Diaz, Jaramillo, Pardo (AIHPB, 2022): Gaussian fluctuations

Proof uses

- CLT in the Wiener chaos (related to Fourth Moment Problem of Nualart & Peccati (Ann Probab, 2005))
- Free independence and multiple Wigner integrals
- Weak convergence results
- Stein methods
- Malliavan Calculus

Part III.B: From Fractional Wishart process to Non-commutatitve Wishart process

Extensions of free Wishart process

III.B. Fractional Wishart process

- $m, n \geq 1$, $m \times n$ matrix process

$$\{B_{m,n}(t)\}_{t\geq 0} = \left\{ \left(b_{m,n}^{j,k}(t) \right)_{1\leq j\leq m, 1\leq k\leq n} \right\}_{t\geq 0},$$

  $\left\{ \mathrm{Re}\left(b_{m,n}^{j,k}(t) \right) \right\}_{t\geq 0}$ & $\left\{ \mathrm{Im}\left(b_{m,n}^{j,k}(t) \right) \right\}_{t\geq 0}$ independent 1-dimensional fractional Bm of parameter $H \in [1/2, 1)$.

- Fractional Laguerre, fractional Wishart process: $n \times n$ matrix-valued process

$$L_{m,n}(t) = B_{m,n}^*(t) B_{m,n}(t), \; t \geq 0$$

- $0 \leq \lambda_n(t) \leq \cdots \leq \lambda_1(t)$ eigenvalues of $L_{m,n}(t)/n$
- For $H \in [1/2, 1)$ the noncoliding property holds

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > \ldots > \lambda_n(t) > 0 \quad \forall t > 0 \right) = 1$$

III.B. Fractional Wishart process

- $H = 1/2$:
  - Bru (JMA, 1989): noncoliding property and stochastic dynamics

$$\mathrm{d}\lambda_i(t) = \lambda_i(0) + \frac{1}{\sqrt{n}}\sqrt{2\lambda_i(t)}W_i(t)$$

$$+\frac{1}{n}\int_0^t \left(m + \sum_{j\neq i}\frac{\lambda_i(s)+\lambda_j(s)}{\lambda_i(s)-\lambda_j(s)}\right)\mathrm{d}s,\ 1 \leq i \leq n$$

  - Cabanal-Duvillard & Guionnet (AP, 2001), Pérez-Abreu & Tudor (EJP, 2009): limiting measure-valued process, when $n/m \to c > 0$, is dilation of free Poisson law
- $H \in (1/2, 1)$: Pardo, Pérez G., Pérez-Abreu (JFA, 2017):
  - Noncoliding, stochastic dynamics of eigenvalues
  - Limiting measure valued process is fractional dilation of free Poisson law

III.B. Dilation rather than the law of free Poisson

Law of non-commutatitve fractional Wishart process

The limit, when $n/m \to c > 0$, of $\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j(t)}$, $t \geq 0$,

- *is not the law of a Free Poisson process* $\{\mathrm{m}_{ct,b}\}_{t\geq 0}$,

$$\mathrm{m}_{a,b}(\mathrm{d}x) = \begin{cases} f_{a,b}(x)\mathrm{d}x, & a \geq 1 \\ (1-a)\delta_0(\mathrm{d}x) + f_{a,b}(x)\mathrm{d}x, & 0 \leq a < 1, \end{cases}$$

$$f_{a,b}(x) = \frac{1}{2\pi bx}\sqrt{4ab^2 - (x - b(1+a))^2}\mathbf{1}_{[b(1-\sqrt{a})^2, b(1+\sqrt{a})^2]}(x)$$

- *rather fractional dilations* of $\mathrm{m}_{c,1}$: $\mu_c^H(t) = \mathrm{m}_{c,1} \circ h_t^{-1}$, for $h_t(x) = t^{2H}x$, i.e.

$$\mu_c^H(t) = \mathrm{m}_{c,t^{2H}}$$

III.B. Characterization of the law

Cabanal-Duvillard & Guionnet (AP, 2001): $H = 1/2$

Pardo, Pérez G, Pérez-Abreu (JFA, 2017): $H \in (1/2, 1)$

Theorem

The family $(\mu_c^H(t), t \geq 0)$ is characterized by the property that its Cauchy transform $G_{c,H}$ is the unique solution to

$$\frac{\partial G_{c,H}}{\partial t}(t,z) = 2Ht^{2H-1}\left[\begin{array}{c} G_{c,H}^2(t,z) + \\ (1-c+2zG_{c,1/2}(t,z))\frac{\partial G_{c,H}}{\partial z}(t,z) \end{array}\right], t > 0$$

$$G_{c,H}(0,z) = \int_{\mathbb{R}} \frac{\mu_{c,H}(0)(dx)}{x-z}$$

Extensions: Wishart matrices built from independent 1-dimensional SDE driven by fBm, $H > 1/2$ (Song, Yao, Yuan, 2021).

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