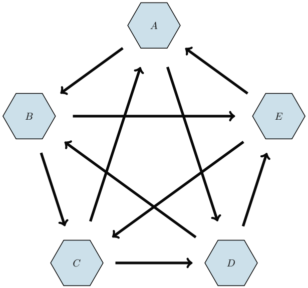
A Central Limit Theorem for intransitive dice



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Jornadas de pesquisa para graduação

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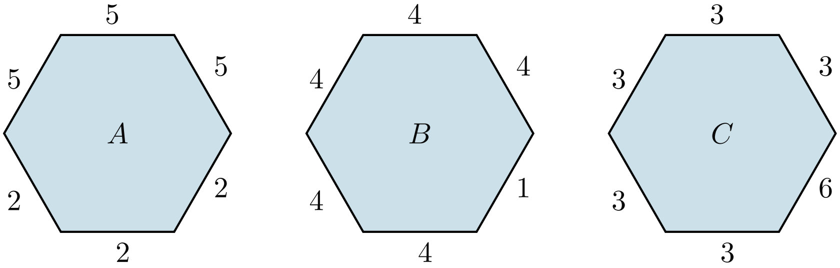
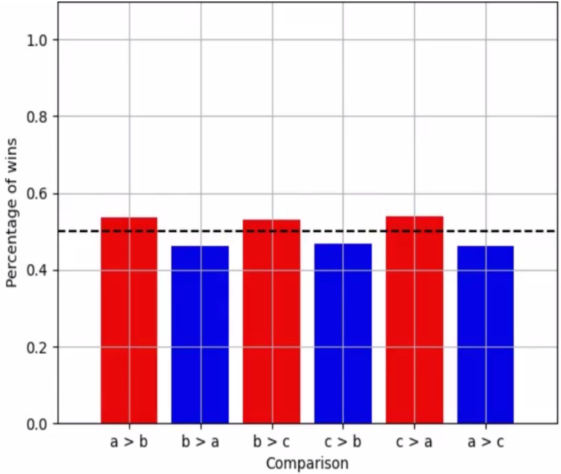
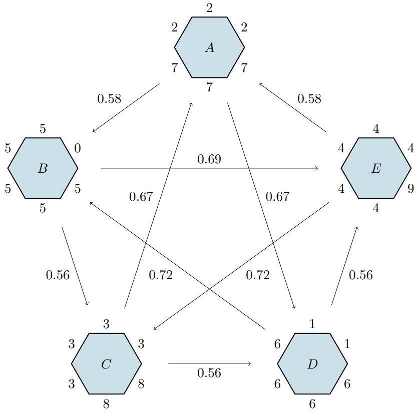
Intransitive Dice

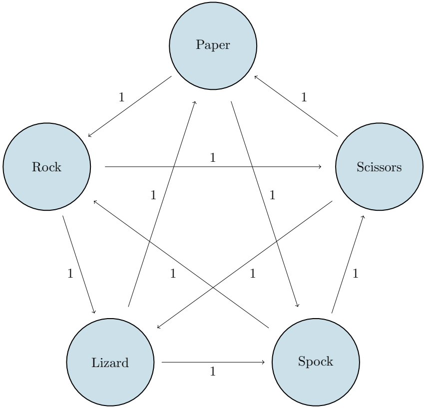
Which die is better?




1.0
0.8
0.6
0.4
0.2
0.0
Percentage of wins
a > b
b > a
b > c
c > b
c > a
a > c
Comparison

Rock, paper, scissors?





- It looks counter-intuitive

However, it is related to ‘real world problems’:

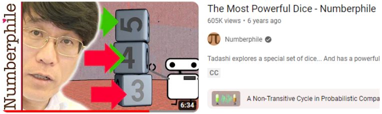
- Testing different drugs;

- Comparing soccer teams;

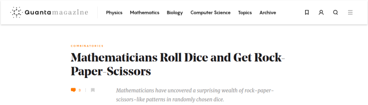
Motivation

- Ludic aspect / popularization of math

- Numberphile (YouTube channel)



- Quanta Magazine (2023)



- Interesting math
  - Polymath Project

THE PROBABILITY THAT A RANDOM TRIPLE OF DICE IS TRANSITIVE

D. H. J. POLYMATH

Abstract. An *n-sided die* is an n-tuple of positive integers. We say that a die $(a_1, \ldots, a_n)$ *beats* a die $(b_1, \ldots, b_n)$ if the number of pairs (i, j) such that $a_i > b_j$ is greater than the number of pairs (i, j) such that $a_i < b_j$. We show that for a natural model of random n-sided dice, if A, B and C are three random dice then the probability that A beats C given that A beats B and B beats C is approximately 1/2. In other words, the information that A beats B and B beats C has almost no effect on the probability that A beats C. This proves a statement that was conjectured by Conrey, Gabbard, Grant, Liu and Morrison for a different model.

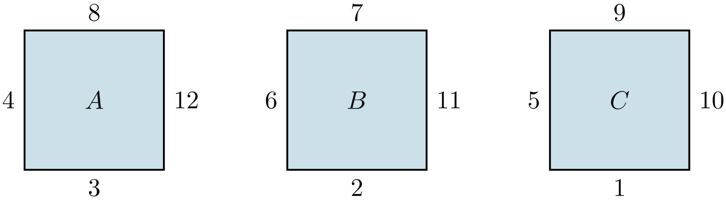
1. Introduction

Some questions

- Deterministic conditions for intransitivity?
- What is the probability that a random set of dice is intransitive?
- How many sets of dice with intransitive cycles are there?

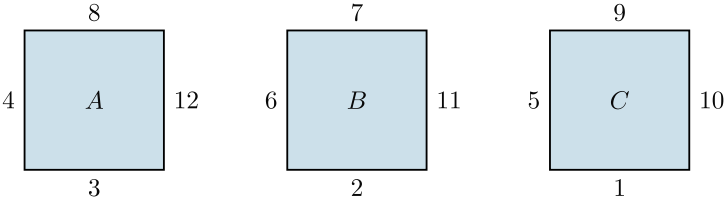
Deterministic conditions

Checking intransitivity – honest dice, no ties



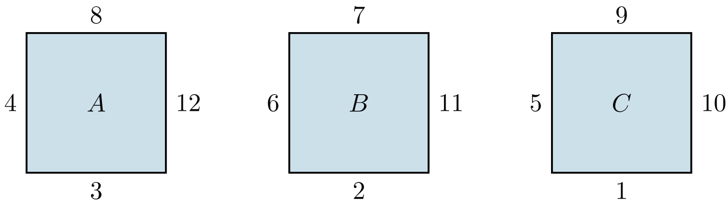
$\boldsymbol{A}$												
$\boldsymbol{B}$												
$\boldsymbol{C}$												

Checking intransitivity – honest dice, no ties



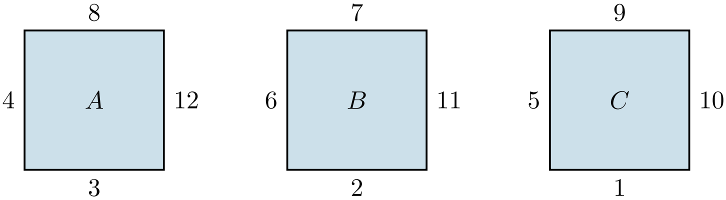
A	12											
B												
C												
	A											

Checking intransitivity – honest dice, no ties



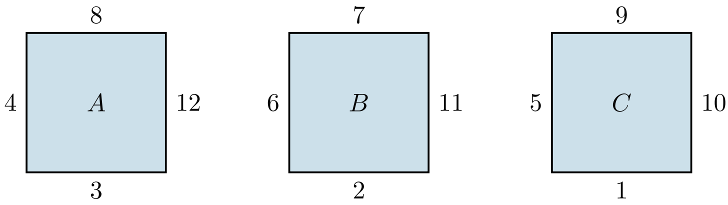
A	12											
B		11										
C												
	A	B										

Checking intransitivity – honest dice, no ties



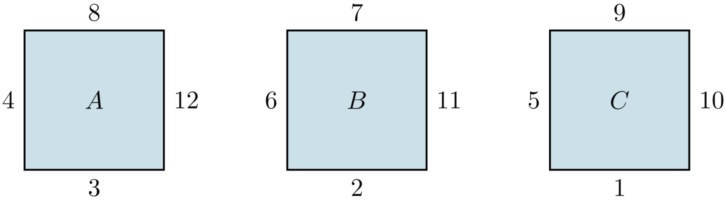
A	12											
B		11										
C			10									
	A	B	C									

Checking intransitivity – honest dice, no ties



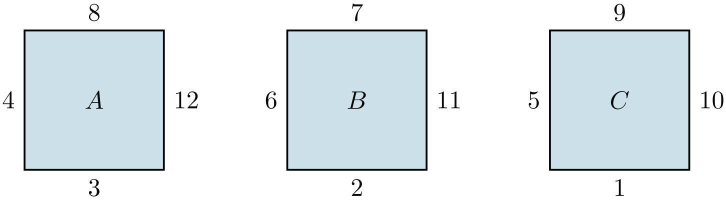
A	12											
B		11										
C			10	9								
	A	B	C	C								

Checking intransitivity – honest dice, no ties

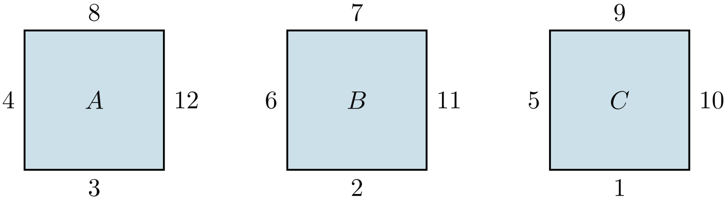


A	12				8							
B		11										
C			10	9								
	A	B	C	C	A							

Checking intransitivity – honest dice, no ties



A	12				8				4	3		
B		11				7	6				2	
C			10	9				5				1
	A	B	C	C	A	B	B	C	A	A	B	C



A	12				8				4	3		
B		11				7	6				2	
C			10	9				5				1
	A	B	C	C	A	B	B	C	A	A	B	C

Without ties: intransitivity $\iff$ counting strings

Existence of intransitive dice

- Die with n faces: $D = (D_1, \ldots, D_n)$;
- Collection of ℓ dice: $\mathbf{D} = (D^{(1)}, \ldots, D^{(\ell)})$;
- When there are no ties: $D^{(i)}$ has n_i faces $\implies$ total of $n_1 + \ldots + n_\ell$ faces;
- Associated string: a permutation of $n_1 + \ldots + n_\ell$ letters (with repetitions).
- $\mathcal{D}_\ell(n) = \{\mathbf{D} = (D^{(1)}, \ldots, D^{(\ell)});\ D^{(i)} \in \mathbb{Z}^n,\ \{D_i^{(j)}\}_{i,j} = [\ell n]\}$.
- $\mathcal{D}_{\triangleright,\ell}(n) = \{\mathbf{D} = (D^{(1)}, \ldots, D^{(\ell)}); D^{(1)} \triangleright \ldots \triangleright D^{(\ell)} \triangleright D^{(1)}\}$.

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Theorem (Existence)

(i) *For every* $\ell \geq 3$, *we have* $\mathcal{D}_{\triangleright,\ell}(2) = \varnothing$.

(ii) *For every* $\ell \geq 3$ *and* $n \geq 3$, *we have* $\mathcal{D}_{\triangleright,\ell}(n) \neq \varnothing$.

Existence of intransitive dice – proof of (i)

We have $n = 2$. Whenever $D^{(j)} \triangleright D^{(k)}$, the substring on $D^{(j)}$ and $D^{(k)}$ is

$$D^{(j)} D^{(j)} D^{(k)} D^{(k)} \quad \text{or} \quad D^{(j)} D^{(k)} D^{(j)} D^{(k)}.$$

In any case, there is a $D^{(j)}$ to the left of the two copies of $D^{(k)}$.

$$\begin{aligned} D^{(1)} \triangleright \ldots \triangleright D^{(\ell)} \triangleright D^{(1)} \qquad &\Longrightarrow \text{ a } D^{(1)} \text{ to the left of both } D^{(k)} \text{ for } k \in [\ell], \\ &\Longrightarrow \text{ no } D^{(\ell)} \text{ to the left of both } D^{(1)}, \\ &\Longrightarrow D^{(\ell)} \not\triangleright D^{(1)}. \end{aligned}$$

Lemma (More dice)

If $\mathcal{W}_{\triangleright,\ell}(n)$ is non-empty, then $\mathcal{W}_{\triangleright,\ell+1}(n)$ is non-empty.

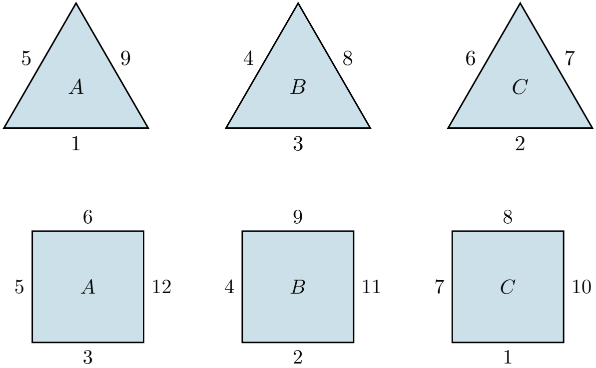
- From $\mathbf{W} \in \mathcal{W}_{\triangleright,\ell}(n)$ make $\widetilde{\mathbf{W}} \in \mathcal{W}_{\triangleright,\ell+1}(n)$ by replacing every $D^{(\ell)}$ by $D^{(\ell)}D^{(\ell+1)}$.

Lemma (More faces)

If $\mathcal{W}_{\triangleright,\ell}(n)$ is non-empty, then $\mathcal{W}_{\triangleright,\ell}(n+2)$ is non-empty.

- For a word $\mathbf{W}$, its *dual word* is $\mathbf{W}^*$ obtained by reversing the order in $\mathbf{W}$;
- Number of wins of $D^{(j)}$ over $D^{(k)}$ is $N_{j,k}(\mathbf{W})$;
- Word $\mathbf{W}$ is *neutral* if $N_{k,j}(\mathbf{W}) = N_{j,k}(\mathbf{W})$ for every j, k.
- If $\mathbf{S} = D^{(1)} \dots D^{(\ell)}$ then the concatenation $\mathbf{SS}^* \in \mathcal{W}_\ell(2)$ is neutral.
- Given any word $\mathbf{W} \in \mathcal{W}_\ell(n)$, the concatenation $\mathbf{WSS}^* \in \mathcal{W}_\ell(n+2)$ is neutral.

Initial cases:



Counting intransitive sets

Question: Can we estimate $|\mathcal{D}_{\triangleright,\ell}(n)|$?

- A simple combinatorial argument and Stirling's approximation ensures

$$|\mathcal{D}_{\triangleright,\ell}(n)| \leq |\mathcal{D}_{\ell}(n)| = \frac{(\ell n)!}{(n!)^{\ell}} \sim \frac{\ell^{1/2}}{(2\pi n)^{(\ell-1)/2}} \cdot e^{n\ell \log \ell}.$$

- Does $|\mathcal{D}_{\triangleright,\ell}(n)|$ grow exponentially?

Theorem

For each $\ell \geq 3$, there exists a constant $L(\ell) \geq 0$ for which

$$|\mathcal{D}_{\triangleright,\ell}(n)| \;=\; e^{nL(\ell)+o(n)} \quad \text{as } n \to \infty\,.$$

Take $\mathbf{W}_1 \in \mathcal{W}_{\triangleright,\ell}(n_1)$ and $\mathbf{W}_2 \in \mathcal{W}_{\triangleright,\ell}(n_2)$. Then, $\mathbf{W}_1\mathbf{W}_2 \in \mathcal{W}_{\triangleright,\ell}(n_1+n_2)$.

$$\begin{aligned} N_{i,i+1}(\mathbf{W}_1\mathbf{W}_2) &= N_{i,i+1}(\mathbf{W}_1) + n_1 n_2 + N_{i,i+1}(\mathbf{W}_2) \\ &> \frac{n_1^2}{2} + \frac{2n_1n_2}{2} + \frac{n_2^2}{2} \\ &= \frac{(n_1+n_2)^2}{2}. \end{aligned}$$

$(\mathbf{W}_1, \mathbf{W}_2) \mapsto \mathbf{W}_1\mathbf{W}_2$ is injection of $\mathcal{W}_{\triangleright,\ell}(n_1) \times \mathcal{W}_{\triangleright,\ell}(n_2)$ into $\mathcal{W}_{\triangleright,\ell}(n_1+n_2)$

$$\Longrightarrow \quad |\mathcal{D}_{\triangleright,\ell}(n_1+n_2)| \geq |\mathcal{D}_{\triangleright,\ell}(n_1)||\mathcal{D}_{\triangleright,\ell}(n_2)|,$$

hence $|\mathcal{D}_{\triangleright,\ell}(n)|$ is *supermultiplicative*! Use Fekete's lemma. □

On the number of intransitive words – value of $L(\ell)$

By the previous theorem, we have

$$L(\ell) = \sup_n \frac{\log|\mathcal{D}_{\triangleright,\ell}(n)|}{n} \leq \sup_n \frac{\log|\mathcal{D}_{\ell}(n)|}{n} = \ell \log \ell.$$

By computational methods we got:

- Exact computations:

$$2.445 < L(3) \leq 3 \log 3.$$

- Simulations for larger values of n:

$$\Delta L_3(n) := \frac{\log|\mathcal{D}_{\triangleright}(n)|}{n} - \frac{\log|\mathcal{D}_{\triangleright,\ell}(n)|}{n} \quad \text{seems to go to zero.}$$

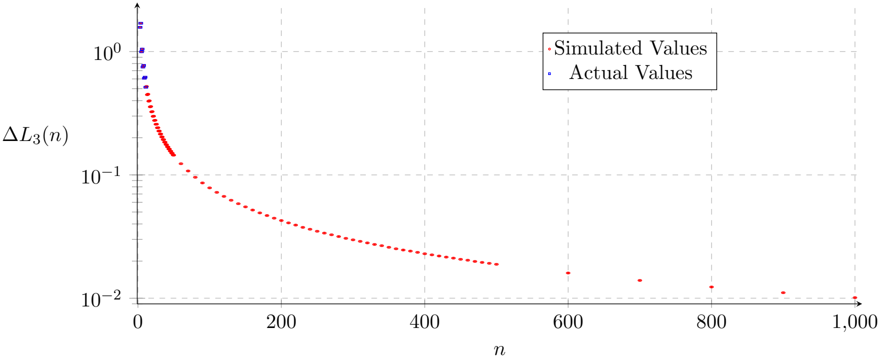


Figure: $\Delta L_3(n)$ for various values of n. The blue data points are exact, and the red data points were generated through a stochastic simulation. The vertical axis is represented in a logarithmic scale

On our arxiv version of the paper:

- Simulations lead to conjecture that $L(3) = 3\log 3$.
- Maybe the same would hold for $\ell \geq 4$.

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Currently unpublished:

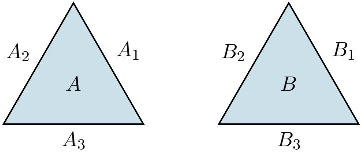
Theorem

For $\ell \geq 3$ it holds that $L(\ell) = \ell \log \ell$.

- Application of our CLT!

Random sets of dice

A and B are n-faced IID dice:



$$N_{A>B} = \sum_{i=1}^{n} \sum_{j=1}^{n} \chi_{A_i>B_j}$$

$$N_{A>B} + N_{B>A} = n^2$$

IID Dice – no ties

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$$A \triangleright B \quad \Longleftrightarrow \quad N_{A>B} > \frac{n^2}{2}$$

IID Dice – no ties

$$N_{A>B} + N_{B>A} = n^2$$

$$A \triangleright B \quad \iff \quad N_{A>B} > \frac{n^2}{2}$$

Assuming (A_i) and (B_j) have a common distribution:

$$\mathbb{E}N_{A>B} = \sum_{i,j=1}^{n} \mathbb{P}(A_i > B_j) = \frac{n^2}{2}$$

$$N_{A>B} + N_{B>A} = n^2$$

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$$A \rhd B \iff N_{A>B} > \mathbb{E}N_{A>B}$$

Hence:

$$A \triangleright B \quad \iff \quad \tilde{N}_{A>B} = \frac{N_{A>B} - \mathbb{E}N_{A>B}}{(\operatorname{Var} N_{A>B})^{1/2}} > 0$$

Hence:

$$A \rhd B \quad \Longleftrightarrow \quad \tilde{N}_{A>B} = \frac{N_{A>B} - \mathbb{E}N_{A>B}}{(\mathrm{Var}\, N_{A>B})^{1/2}} > 0$$

If we have $\ell \geq 3$ dice:

$$D^{(1)} \rhd D^{(2)} \rhd \ldots \rhd D^{(\ell)} \rhd D^{(1)}$$
$$\Updownarrow$$
$$(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_\ell) \in (0, \infty)^\ell$$

Main question: As $n \to \infty$,

$$(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_\ell) \xrightarrow{d} \quad ?$$

Theorem (Uniform case, $\ell = 3$)

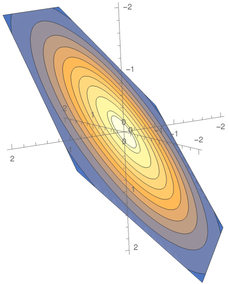
Suppose (A_i), (B_i), (C_i) *are iid. with distribution* $\mathrm{Unif}(0,1)$.

$$(\tilde{N}_{A>B}, \tilde{N}_{B>C}, \tilde{N}_{C>A}) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

where

$$\Sigma = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

IID Dice – asymptotic transitivity



Plane $x + y + z = 0$

Matrix Σ is singular:

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Corollary (Uniform case, $\ell = 3$)

$$\mathbb{P}(A \triangleright B \triangleright C \triangleright A) \xrightarrow{n} \mathbb{P}(\mathcal{N}(0, \Sigma) \in (0, \infty)^3) = 0.$$

$\mathcal{D}_3(n)$: 3n-letter strings, with n letters A, B and C.

$\mathcal{D}_{\triangleright,3}(n)$: intransitive strings in $\mathcal{D}_3(n)$.

Corollary (Uniform case, $\ell = 3$) $\implies$ $\dfrac{|\mathcal{D}_{\triangleright,3}(n)|}{|\mathcal{D}_3(n)|} \to 0.$

Proof ideas – method of moments

$$\tilde{N}_{A>B} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\Updownarrow$$

$$\mathbb{E}[(\tilde{N}_{A>B})^t] \to \mathbb{E}[\mathcal{N}(0,1)^t] = \begin{cases} 0 & \text{if } t \text{ is odd} \\ (t-1)!! & \text{if } t \text{ is even} \end{cases},$$

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$$\operatorname{Var} N_{A>B} = n^3\sigma + O(n^2)$$

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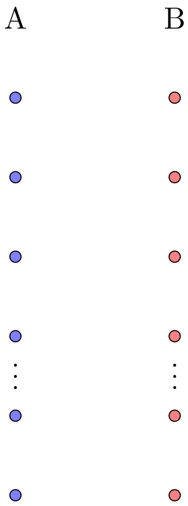
$$\operatorname{Var} N_{A>B} = n^3\sigma + O(n^2)$$

$$\tilde{N}_{A>B} = \frac{N_{A>B} - \mathbb{E}N_{A>B}}{(\operatorname{Var} N_{A>B})^{1/2}} \quad \implies \quad \mathbb{E}[(\tilde{N}_{A>B})^t] = \frac{\mathbb{E}\left[\left(\sum_{i,j}^n \overbrace{\left(\chi_{A_i>B_j} - \frac{1}{2}\right)}^{e_{i,j}}\right)^t\right]}{n^{3t/2}(\sigma + o(1))}$$

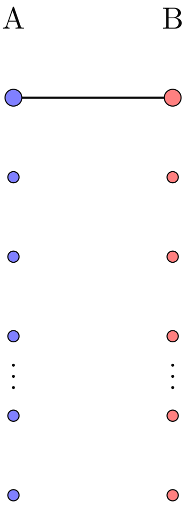
Proof ideas – using graphs



$$\mathbb{E}\Big[\Big(\sum_{i,j} e_{i,j}\Big)^t\Big] = \sum_{i_1,j_1} \cdots \sum_{i_t,j_t} \mathbb{E}[e_{i_1,j_1} \dots e_{i_t,j_t}]$$

$$e_{1,1}\ e_{2,3}\ e_{2,1}\ \dots\ e_{i_t,j_t}$$

Proof ideas – using graphs



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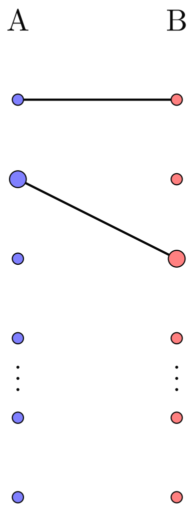

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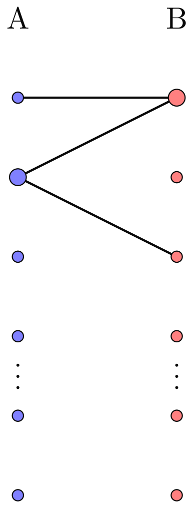

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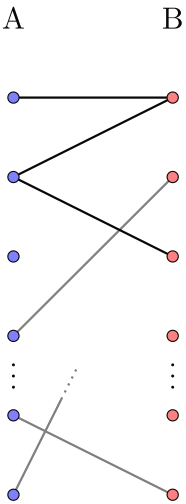

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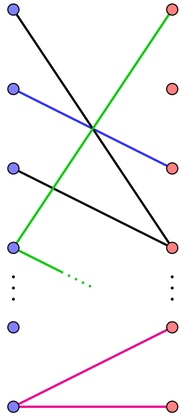
Proof ideas – using graphs



$$\mathbb{E}\left[\left(\sum_{i,j} e_{i,j}\right)^t\right] = \sum_{i_1,j_1} \cdots \sum_{i_t,j_t} \mathbb{E}[e_{i_1,j_1} \dots e_{i_t,j_t}]$$

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- Connected components ↔ independence

$$\mathbb{E}[\underbrace{(e_{i_1,j_1} \dots e_{i_{t_1},j_{t_1}})}_{G_1} \dots G_k] = \prod_{i=1}^{k} \mathbb{E}[G_i]$$

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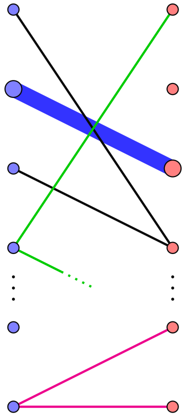

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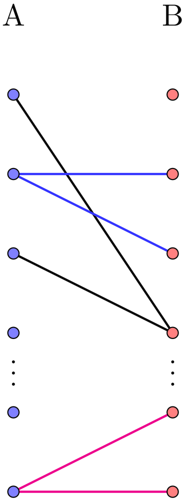
- Isolated edges:

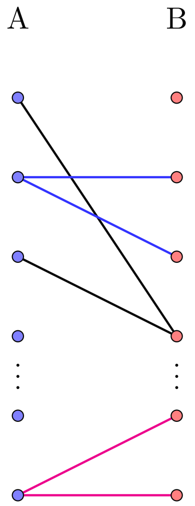
$$\mathbb{E}[e_{i,j}] = \mathbb{P}(A_i > B_j) - \tfrac{1}{2} = 0.$$

Proof ideas – counting cherries



Sufficient to count cherry graphs!

Proof ideas – counting cherries



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- More than $t/2$ connected components:

  At least one with one edge.

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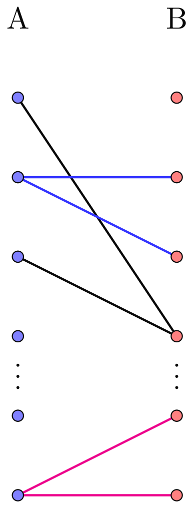

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Since $|\mathbb{E}[G]|$ is bounded, can be ignored.

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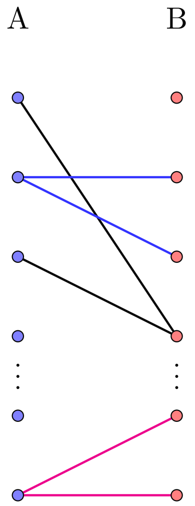

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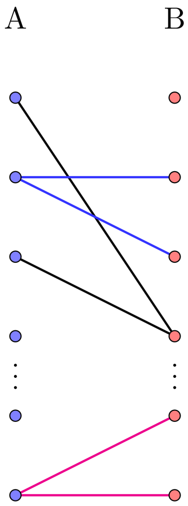

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Proof ideas – convergence of random vector

By the Cramér-Wold criteria:

$$(\tilde{N}_{A>B}, \tilde{N}_{B>C}, \tilde{N}_{C>A}) = (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

$$\Updownarrow$$

For every $\alpha = (\alpha_1, \alpha_2, \alpha_3)$:

$$\sum_{i=1}^{3} \alpha_i \tilde{N}_i \xrightarrow{d} \sum_{i=1}^{3} \alpha_i X_i, \text{ where } X = (X_1, X_2, X_3)^T \sim \mathcal{N}(0, \Sigma).$$

$$\Updownarrow$$

For every $\alpha = (\alpha_1, \alpha_2, \alpha_3)$:

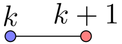
$$\mathbb{E}\Big[\Big(\sum_{i=1}^{3} \alpha_i \tilde{N}_i\Big)^t\Big] \to \mathbb{E}\Big[\Big(\sum_{i=1}^{3} \alpha_i X_i\Big)^t\Big] = \begin{cases} 0 & \text{if } t \text{ is odd} \\ (\alpha^T \Sigma \alpha)^{t/2} (t-1)!! & \text{if } t \text{ is even} \end{cases},$$

CLT – General case

We allow quite general sequence of dice!

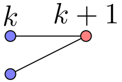
- $\ell \geq 3$ dice: $\mathbf{D} = (D^{(1)}, D^{(2)}, \ldots, D^{(\ell)})$;
- We look at the laws of sequence of dice $\{\mathbf{D}_m\}_m$;
- Different laws: $D^{(j)}(m)$ has iid faces with law $\mathcal{L}_m^{(j)}$;
- Different dice are independent;
- Different number of faces: $n_j(m)$;
  (for convenience, $n_j = f_j \cdot m$, with $f_j \leq 1$.)

Single edge:



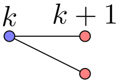
$$\mathbf{p}_k \;=\; \mathbf{p}(\mathcal{L}^{(k)}, \mathcal{L}^{(k+1)}) \;:=\; \mathbb{P}\left(D_1^{(k)} > D_1^{(k+1)}\right) \;=\; \mathbb{E}\left(\mathbb{1}_{D_1^{(k)} > D_1^{(k+1)}}\right)$$

Cherry of type $(k+1, 3)$**:**



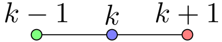
$$\mathbf{q}_k \;=\; \mathbf{q}(\mathcal{L}^{(k)}, \mathcal{L}^{(k+1)}) \;:=\; \mathbb{P}\left(D_1^{(k)} > D_1^{(k+1)}, D_2^{(k)} > D_1^{(k+1)}\right)$$

Cherry of type $(k, 2)$:



$$\mathbf{r}_k = \mathbf{r}(\mathcal{L}^{(k)}, \mathcal{L}^{(k+1)}) := \mathbb{P}\left(D_1^{(k)} > D_1^{(k+1)}, D_1^{(k)} > D_2^{(k+1)}\right)$$

Cherry of type $(k, 1)$:



$$\mathbf{s}_k = \mathbf{s}(\mathcal{L}^{(k-1)}, \mathcal{L}^{(k)}, \mathcal{L}^{(k+1)}) := \mathbb{P}\left(D_1^{(k-1)} > D_1^{(k)} > D_1^{(k+1)}\right)$$

CLT – General case; important quantities

From $\mathbf{p}_k, \mathbf{q}_k, \mathbf{r}_k, \mathbf{s}_k$ we have many important quantities. Let:

$$\sigma_k := \left[f_k f_{k+1} \left(f_k(\mathbf{q}_k - \mathbf{p}_k^2) + f_{k+1}(\mathbf{r}_k - \mathbf{p}_k^2) \right) \right]^{1/2}$$
$$\gamma_k := \frac{1}{\sigma_{k-1}\sigma_k} f_{k-1} f_k f_{k+1} (\mathbf{s}_k - \mathbf{p}_{k-1}\mathbf{p}_k) \, .$$

Then, we have

$$\begin{aligned} \mathbb{E}(N_k) &= f_k f_{k+1} m^2 \mathbf{p}_k \, , \\ \mathrm{Var}\,(N_k) &= \sigma_k^2 m^3 + o(m^3) \, , \quad \text{and} \\ \mathrm{Corr}\,(N_{k-1}, N_k) &= \gamma_k + o(1) \, . \end{aligned}$$

Assumptions

Fix $\ell \geq 3$. Assume the sequence $\{\mathbf{D}_m\}_m$ satisfies

(i) For $k = 1, \ldots, \ell$ the relative sizes $f_k = f_k(m)$ satisfy

$$f_k(m) \to f_k(\infty) \in (0, 1], \quad \text{as } m \to \infty.$$

(ii) For $k = 1, \ldots, \ell$ the rate of growth of the mean and variance of N_k and covariance between N_{k-1} and N_k satisfy

$$\begin{aligned} \mathbf{p}_k(m) &\to \mathbf{p}_k(\infty) \in (0, 1], \\ \sigma_k(m) &\to \sigma_k(\infty) \in (0, \infty)\,, \\ \gamma_k(m) &\to \gamma_k(\infty) \in [-1, 1]\,, \end{aligned}$$

as $m \to \infty$.

Theorem

Under Assumptions, we have

$$(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_\ell) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \Sigma),$$

where

$$\Sigma = \begin{pmatrix} 1 & \gamma_2(\infty) & 0 & \cdots & 0 & \gamma_1(\infty) \\ \gamma_2(\infty) & 1 & \gamma_3(\infty) & \cdots & 0 & 0 \\ 0 & \gamma_3(\infty) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \gamma_\ell(\infty) \\ \gamma_1(\infty) & 0 & 0 & \cdots & \gamma_\ell(\infty) & 1 \end{pmatrix}.$$

Connecting CLT with intransitivity, ties allowed

No-tie dice: $\mathbb{P}(D^{(1)} \triangleright \ldots \triangleright D^{(\ell)} \triangleright D^{(1)}) \to \mathbb{P}(\mathcal{N}(0, \Sigma) \in (0, \infty)^{\ell})$.

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Theorem (Connecting CLT with intransitivity, ties allowed)

Fix $\ell \geq 3$ and assume $\{\mathbf{D}_m\}_m$ satisfy the Assumptions. Suppose that there exists $\delta > 0$ and a function $r(m)$ with $\lim_{m\to\infty} r(m) = +\infty$, for which

$$\frac{1}{2} - \mathbf{p}_k - \frac{1}{2}\mathbb{P}(D_1^{(k)} = D_1^{(k+1)}) \;\geq\; -\frac{\delta}{m^{1/2} r(m)}\,, \quad k = 1,\ldots,\ell,$$

for every m sufficiently large, and in addition

$$\lim_{m\to\infty} \mathbb{P}\left(D_1^{(k)}(m) = D_1^{(k+1)}(m)\right) \;=\; 0 \quad \text{for } k = 1,\ldots,\ell.$$

Then

$$\limsup_{m\to\infty} \mathbb{P}\left(D^{(1)} \triangleright \cdots \triangleright D^{(\ell)} \triangleright D^{(1)}\right) \;\leq\; \mathbb{P}\left(\mathcal{N}(0,\Sigma) \in [0,\infty)^{\ell}\right).$$

Important case: all laws are the same

- $\frac{1}{2} - \mathbf{p}_k - \frac{1}{2}\mathbb{P}(D_1^{(k)} = D_1^{(k+1)}) = 0$ always holds.
- We have that asymptotically, when probability of ties go to zero:

$$\mathbf{p}_k \to \frac{1}{2}, \quad \mathbf{q}_k \to \frac{1}{3}, \quad \mathbf{r}_k \to \frac{1}{3}, \quad \mathbf{s}_k \to \frac{1}{6}$$

- Consequently:

$$\sigma_k(\infty) = \sqrt{\frac{f_k f_{k+1}(f_k + f_{k+1})}{6}}$$

$$\gamma_k(\infty) = -\frac{f_{k-1} f_k f_{k+1}}{\sqrt{f_{k-1} f_k (f_{k-1} + f_k)}\sqrt{f_k f_{k+1}(f_k + f_{k+1})}}$$

IID case with ties going to zero

- Covariance matrix Σ has a very interesting structure!

Lemma

In iid. case with probability of ties going to zero, we have that $\det \Sigma = 0$ *and the eigenspace of* 0 *has dimension 1 and is generated by a vector* $x \in (0, \infty)^\ell$.

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Corollary

In iid. case with probability of ties going to zero:

$$\mathbb{P}\left(D^{(1)} \triangleright \cdots \triangleright D^{(\ell)} \triangleright D^{(1)}\right) \to \mathbb{P}\left(\mathcal{N}(0, \Sigma) \in [0, \infty)^{\ell}\right) = 0.$$

Example: intransitive cycles with high probability

Theorem

Let $A^{(k)} = (a_1^{(k)}, \ldots, a_m^{(k)})$ for $k \in [\ell]$ be a set of ℓ deterministic honest dice with m faces that is known to be intransitive: $A^{(k)} \triangleright A^{(k+1)}$ for every k. Consider random dice $(B^{(k)} : k \in [\ell])$, each with n faces, where the faces of die $B^{(k)}$ are independently chosen with law

$$B_j^{(k)} \sim \mathrm{Unif}\{a_i^{(k)} : i \in [m]\},$$

that is, uniformly over the faces of die $A^{(k)}$. Then, there is a constant $c > 0$, depending only on the set of dice $A^{(k)}$, such that

$$\mathbb{P}\big(B^{(1)} \triangleright B^{(2)} \triangleright \cdots \triangleright B^{(\ell)} \triangleright B^{(1)}\big) = 1 + o(e^{-cn}), \quad \text{as } n \to \infty. \tag{1}$$

Exponential rate of growth of intransitive words

Theorem

Fix $\ell \geq 3$. It holds $\frac{\log |\mathcal{D}_{\triangleright,\ell}(n)|}{n} \to \ell \log \ell$.

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- We saw $\frac{|\mathcal{D}_{\triangleright,\ell}(n)|}{|\mathcal{D}_{\ell}(n)|} \to 0$.
- Let $\mathcal{Q}_\ell(n)$ be a set of close to intransitive words:

$$\mathcal{Q}_\ell(n) := \Big\{\mathbf{W} \in \mathcal{W}_\ell(n);\ 2N_{i,i+1} > n^2 - n^{3/2}\Big(1 + \frac{1}{2n}\Big)^{1/2}\Big\}$$
$$= \{(\tilde{N}_1, \ldots, \tilde{N}_\ell) \in (-\frac{\sqrt{6}}{2}, \infty)\}.$$

  Then, the CLT implies $\frac{|\mathcal{Q}_\ell(n)|}{|\mathcal{D}_\ell(n)|} \to c > 0$.
  Hence, $|\mathcal{Q}_\ell(n)|$ has exponential growth of order $\ell \log \ell$.
- Build intransitive words by concatenating to $\mathbf{W} \in \mathcal{Q}_\ell(n)$ a highly intransitive word.

Obrigado!