Properties of the (fermionic) gradient squared of the Gaussian free field

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Section 1

Motivation

Abelian sandpile model (ASM)

The height-one field

1. Choose at time zero a function $s : \Lambda \to \mathbb{N}$, $\Lambda \Subset \mathbb{Z}^{\mathrm{d}}$

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4. If $s(x) > 2d$ (instability), topple x sending one "grain" to each neighbor

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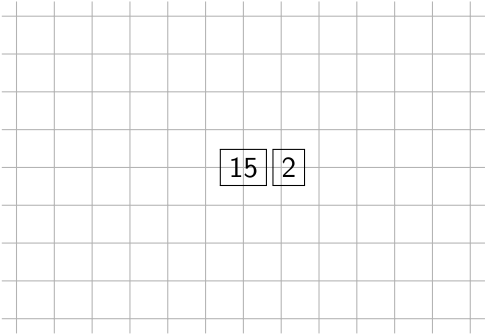
This Markov chain has a unique stationary measure $\mathbb{P}$.

Definition (Height-one field)

$h_\Lambda(x) := \mathbf{1}_{\{s(x)=1\}}$ under $\mathbb{P}$

An example

$$s(x) = 15\,\delta_{x=0} + 2\,\delta_{x=(1,0)}$$



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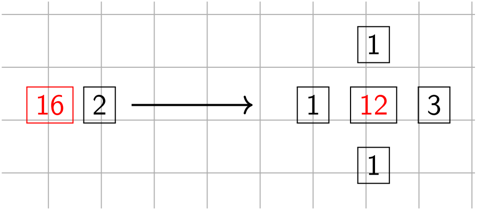

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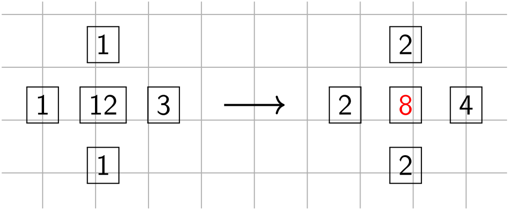

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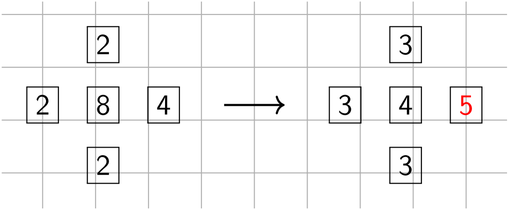

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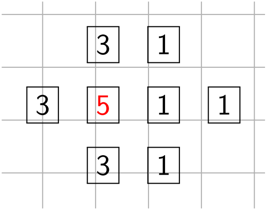

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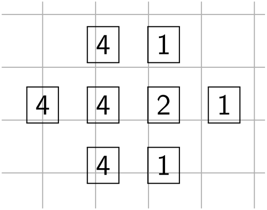

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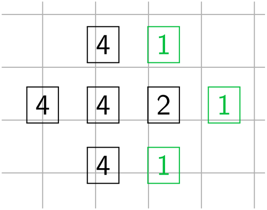

Stable configuration!

Abelian sandpile

Ingredients

- Let $U \subset \mathbb{R}^2$ be smooth connected bounded and $\Lambda := U_\epsilon := U/\epsilon \cap \mathbb{Z}^2$

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- Let $g_U(\cdot, \cdot)$ be the harmonic Green's function on U with Dirichlet boundary conditions
- Joint cumulants κ for r. v.'s $X_1, \ldots, X_n$ are defined by

$$E\left[\prod_{i=1}^{n} X_i\right] = \sum_{\pi \text{ partition of } \{1, \ldots, n\}} \prod_{B \in \pi} \kappa(X_i : i \in B)$$

Eg. $\kappa(X) = E[X]$, $\kappa(X, Y) = \text{cov}(X, Y)$.

Abelian sandpile

Height-one field in $d = 2$

Theorem (Dürre (2009))

Theorem 2 (*Scaling Limit for the Height One Joint Cumulants*). *Let V be as in Theorem 1 and suppose $|V| \geq 2$. Then as $\epsilon \to 0$ the rescaled joint cumulant $\epsilon^{-2|V|}\kappa\left(h_{U_\epsilon}(v_\epsilon) : v \in V\right)$ converges to*

$$\kappa_U(v : v \in V) := -C^{|V|} \sum_{\sigma \in S_{\mathrm{cycl}}(V)} \sum_{(k^v)_{v\in V} \in \{x,y\}^V} \prod_{v \in V} \partial^{(1)}_{k^v} \partial^{(2)}_{k^{\sigma(v)}} g_U\left(v, \sigma(v)\right).$$

Here $C := (2/\pi) - (4/\pi^2)$. That is, if we write $\kappa_U(v) := 0$ for all $v \in V$, then

$$\lim_{\epsilon \to 0} \epsilon^{-2|V|} \mathbb{E}\left[\prod_{v \in V}\left(h_{U_\epsilon}(v_\epsilon) - \mathbb{E}[h_{U_\epsilon}(v_\epsilon)]\right)\right] = \sum_{\Pi \in \Pi(V)} \prod_{B \in \Pi} \kappa_U(v : v \in B).$$

Abelian sandpile

The connection to GFF

Let Ψ be a Gaussian free field with 0-boundary conditions:

Definition (GFF)

Ψ is the centered Gaussian random distribution with

$$\mathbb{E}[\Psi(u)\Psi(v)] = g_U(u, v), \quad u \neq v \in U.$$

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$$: \|\nabla \Psi(v)\|^2 := \sum_{i=1}^{2} \partial_i \Psi(v)^2 - \mathbb{E}[\partial_i \Psi(v)^2]$$

Section 2

Model 1

Grad squared DGFF

Definition (DGFF)

Let $(\Gamma_\epsilon(v) : v \in U_\epsilon)$ be the discrete GFF on U_ϵ:

$$\mathbb{E}[\Gamma_\epsilon(v)] = 0, \quad \mathbb{E}[\Gamma_\epsilon(v)\Gamma_\epsilon(u)] = G_{U_\epsilon}(u, v)$$

where $G_{U_\epsilon}(\cdot, \cdot)$ is the discrete harmonic Green's function with Dirichlet b.c.

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Definition (Grad squared DGFF)

The field $(\Phi_\epsilon(v), v \in U_\epsilon)$ is defined as

$$\Phi_\epsilon(v) := \sum_{i=1}^{d} : \nabla_i \Gamma_\epsilon(x)^2 := \sum_{i=1}^{d} : (\Gamma_\epsilon(v + e_i) - \Gamma_\epsilon(v))^2 :$$

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We will work in $d \geqslant 2$

Grad squared DGFF

Covariances

Call $[d] := \{1, \ldots, d\}$.

$$\mathbb{E}\left[\Phi_\epsilon(x_\epsilon)\Phi_\epsilon(y_\epsilon)\right] = 2 \sum_{i,j\in[d]} \left(\nabla_i^{(1)}\nabla_j^{(2)} G_{U_\epsilon}(x_\epsilon, y_\epsilon)\right)^2$$

Section 3

Results model 1

Main results

Convergence of cumulants, $d \geqslant 2$

Theorem (C, Hazra, Rapoport, Ruszel 2022)

Let $\mathcal{E}$ *be the set of coordinate vectors of* $\mathbb{R}^d$. *Let* $\{v^{(1)}, \dots, v^{(k)}\} \subset U$. *Let* $S^0_{\text{cycl}}(B)$ *be the set of cyclic permutations of a set* B. *If* $v^{(i)} \neq v^{(j)}$ *for all* $i \neq j$, *then*

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Let $\mathcal{E}$ be the set of coordinate vectors of $\mathbb{R}^d$. Let $\{v^{(1)}, \ldots, v^{(k)}\} \subset U$. Let $S^0_{\mathrm{cycl}}(B)$ be the set of cyclic permutations of a set B. If $v^{(i)} \neq v^{(j)}$ for all $i \neq j$, then

$$\lim_{\epsilon \to 0} \epsilon^{-dk} \kappa\Big(\Phi_\epsilon\big(v^{(j)}_\epsilon\big) : j \in [k]\Big)$$

$$= 2^{k-1} \sum_{\sigma \in S^0_{\mathrm{cycl}}([k])} \sum_{\eta : [k] \to \mathcal{E}} \prod_{j=1}^{k} \partial^{(1)}_{\eta(j)} \partial^{(2)}_{\eta(\sigma(j))} g_U\big(v^{(j)}_\epsilon, v^{(\sigma(j))}_\epsilon\big)$$

In $d = 2$ the limit is conformally covariant with scale dimension 2

Main results model 1

Comparison in $d = 2$

- Dürre:

$$-C^k \sum_{\sigma \in S^0_{\text{cycl}}([k])} \sum_{\eta:[k]\to\mathcal{E}} \prod_{j=1}^{k} \partial^{(1)}_{\eta(j)} \partial^{(2)}_{\eta(\sigma(j))} g_U\left(v^{(j)}, v^{(\sigma(j))}\right)$$

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- CHRR:

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Main results model 1

Comparison in $d = 2$

Corollary

$$\lim_{\epsilon\to 0} \epsilon^{-2k} \kappa\left(h_{U_\epsilon}(v_\epsilon^{(j)}) : j \in [k]\right) = -2 \lim_{\epsilon\to 0} \epsilon^{-2k} \kappa\left(\frac{C}{2}\Phi_\epsilon(v_\epsilon^{(j)}) : j \in [k]\right)$$

Section 4

Model 2

Discrete Laplacian

Discrete Laplacian: for $i, j \in \Lambda$

$$\Delta_\Lambda(i, j) = \begin{cases} 1 & |i - j| = 1 \\ -2d & i = j \\ 0 & \text{otherwise} \end{cases}$$

Fermionic Gaussian free field

Fermionic variables

Definition (Grassmanian variables)

Let $\{\xi_i, \bar{\xi}_i : i \in \Lambda\}$ be symbols that satisfy

$$\xi_i \xi_j = -\xi_j \xi_i, \quad \xi_i \bar{\xi}_j = -\bar{\xi}_j \xi_i, \quad \bar{\xi}_i \bar{\xi}_j = -\bar{\xi}_j \bar{\xi}_i$$

Fermionic Gaussian free field

fGFF

Definition (fGFF)

For every function F of $\{\xi_i, \bar{\xi}_i\}$ the expectation of F under the fGFF is defined as

$$[F] = \int_{Berezin} \partial_{\bar{\xi}} \partial_{\xi} \, e^{(\xi, (-\Delta_\Lambda) \bar{\xi})} \, F.$$

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Example:

$$[1] = \int \partial_{\bar{\xi}} \partial_{\xi} \, e^{(\xi, (-\Delta_\Lambda)\bar{\xi})} = \det(-\Delta_\Lambda).$$

Fermionic Gaussian free field

fGFF & UST

Let $\mathbf{P}$ be the law of the uniform spanning tree T on Λ.

Fermionic Gaussian free field

fGFF & UST

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Proposition

Let S be any subset of edges of Λ.

$$\mathbf{P}(S \subseteq \mathsf{T}) = \frac{1}{\det(-\Delta_\Lambda)} \left[\prod_{e \in S} \nabla_e \xi \nabla_e \bar{\xi} \right]$$

where

$$\nabla_e \xi = \xi_{e^+} - \xi_{e^-}, \quad \nabla_e \bar{\xi} = \bar{\xi}_{e^+} - \bar{\xi}_{e^-}$$

is the gradient of the fGFF along the edge e.

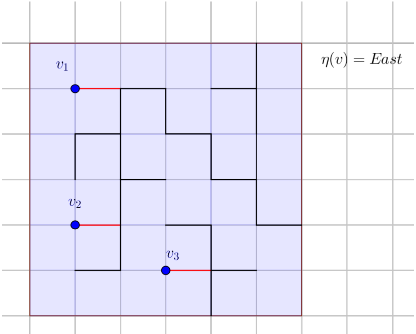
Fermionic Gaussian free field

UST & ASM

Proposition (Dhar–Majumdar, Járai–Werning)

Let $V \subseteq \Lambda$. Let $\eta : V \to [2d]$ be a choice of a direction. Then the height-one field satisfies

$$\mathbb{E}\left(\prod_{v \in V} h_\Lambda(v)\right) = \mathbf{P}((v, v+e) \notin T \text{ if } e \neq \eta(v),\, v \in V).$$



Fermionic Gaussian free field

fGFF & ASM

Now we are able to connect ASM and fGFF via the UST.

- Link height-one $\iff$ fermions conjectured in physics by Jeng, Piroux–Ruelle ('free symplectic fermion theory')

Section 5

Results model 2

Main results

Theorem (Chiarini, C, Rapoport, Ruszel, 2023)

$$\mathbb{E}\left[\prod_{v\in V} h_\Lambda(v)\right] = \frac{1}{\det(-\Delta_\Lambda)}\left[\prod_{v\in V} X_v Y_v\right]$$

where

$$X_v = \sum_{e\ni v} \nabla_e \xi \nabla_e \bar{\xi}$$
$$Y_v = \prod_{e\ni v} \left(1 - \nabla_e \xi \nabla_e \bar{\xi}\right)$$

Proof.

Key: inclusion-exclusion principle over edges. □

Main results

Theorem (CCRR, 2023)

In $\mathrm{d} = 2$

$$\lim_{\epsilon \to 0} \epsilon^{-2k} \kappa\Big(-C_2 X_{v_\epsilon^{(j)}} : j \in [k] \Big)$$
$$= \lim_{\epsilon \to 0} \epsilon^{-2k} \kappa\Big(h_{U_\epsilon}\big(v_\epsilon^{(j)}\big) : j \in [k] \Big), \quad C_2 = 2/\pi - 4/\pi^2.$$

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We also have a closed form expression for the limiting cumulants of $-C_d X_v$ in $d \geqslant 3$.

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Proof.

Wick's theorem for fermionic variables and combinatorics of partitions/permutations. □

Main results

Theorem (CCRR, 2023)

In $\mathrm{d} = 2$

$$\lim_{\epsilon \to 0} \epsilon^{-2k} \kappa\Big(X_{v_\epsilon^{(j)}} Y_{v_\epsilon^{(j)}} : j \in [k] \Big)$$
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Proof.

The proof generalizes that for X_v. It is alternative and independent to Dürre's. Essentially, "Y_v becomes $-C$". □

Extension/1

Universality

Our proofs in the limit $\epsilon \to 0$ hold for the triangular and hexagonal lattice as well:

- translation invariance (homogeneity)
- isotropy

Extension/2

We have on $\mathbb{Z}^d$

$$\kappa\left(X_v,\, v \in V\right) = \kappa\left({}^{\deg_{\mathrm{UST}}(v)}/_{2d},\, v \in V\right).$$

- Analogous results obtained for the "degree field" in $\mathbb{Z}^d$ and triangular lattice
- Other observables of UST can be studied (ongoing)

Thank you!