Multidimensional Stein method and asymptotic independence

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The Stein's method represents a popular probabilistic collection of techniques that allows to evaluate the distances between the probability distributions of random variables.

Given two random variables F, G, the Stein's method allows to obtain sharp estimates for the quantities of the form

$$\sup_{h\in\mathcal{H}} |\mathbf{E}h(F) - \mathbf{E}h(G)|$$

where $\mathcal{H}$ constitutes a large enough class of functions.

Of particular importance is the case when one of the two random variables follows the Gaussian distribution (but the cases of other target distributions are also of interest).

The starting point of the Stein's method for normal approximation is the following observation: $Z \sim N(0, \sigma^2)$ with $\sigma > 0$ if and only if

$$\sigma^2 \mathbf{E} f'(Z) - \mathbf{E} Z f(Z) = 0$$

for every function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{E}|f'(Z)| < \infty$.

Then, one can think that if a random variable X has the property that

$$\sigma^2 \mathbf{E} f'(X) - \mathbf{E} X f(X)$$

is close to zero for a large class of functions f, then the probability distribution of X should be close to $N(0, \sigma^2)$.

From this observation, the whole Stein's theory has been constructed, leading to various bounds for the distance between the probability law of the random variable X and the normal distribution $N(0, \sigma^2)$.

The associated Stein's equation is:

$$\sigma^2 f'(x) - xf(x) = h(x) - \mathbf{E}h(Z),$$

where h is given and satisfies $\mathbf{E}h(Z) < \infty$.

The Stein's equation has a (unique bounded) solution f_h which is differentiable, with f_h' bounded

Then for any random variable F (under suitable assumptions)

$$|\mathbf{E}h(F) - \mathbf{E}h(Z)| = |\sigma^2 \mathbf{E} f_h'(F) - \mathbf{E} F f_h(F)|$$

and

$$\begin{aligned} & \sup_{h \in \mathcal{H}} |\mathbf{E}h(F) - \mathbf{E}h(Z)| = \sup_{h \in \mathcal{H}} |\sigma^2 \mathbf{E} f_h'(F) - \mathbf{E} F f_h(F)| \\ = \quad & \sup_{h \in \mathcal{H}} |\mathbf{E} f_h'(F)(\sigma^2 - \langle DF, D(-L)^{-1} F \rangle)| \\ \leq \quad & C\mathbf{E}|\sigma^2 - \langle DF, D(-L)^{-1} F \rangle|. \end{aligned}$$

Pimentel's work: Let X be a random variable and $\mathbb{Y}$ a random vector in $\mathbb{R}^d$.

Purpose: measure the (Wasserstein) distance between the law pf

$$P_{(X,\mathbb{Y})}$$

and

$$P_Z \otimes P_{\mathbb{Y}},$$

the distribution of a vector with with independent components, where $Z \sim N(0, \sigma^2)$.

The objects from Malliavin calculus

Let $(W_t)_{t\in[0,1]}$ be a standard Wiener process and I_n the multiple integral of order n w.r.t. W.
I_n is an isometry from $L^2[0,1]^n$ onto $L^2(\Omega)$

$$EI_n(f)^2 = n!\|\tilde{f}\|^2_{L^2[0,1]^n}$$

where $\tilde{f}$ is the symmetrization of f

$I_n(f)$ is also an iterated Itô integral
If f is symmetric,

$$I_n(f) = n!\int_0^1 dW_{t_n} \ldots\ldots\ldots \int_0^{t_2} dW_{t_1} f(t_1, \ldots, t_n)$$

Wiener chaos decomposition:
any random variable $F \in L^2(\Omega, \mathcal{F}, P)$ ($\mathcal{F}$ is the sigma-algebra generated by W) can be written as

$$F = \sum_{n \geq 0} I_n(f_n)$$

with $f_n \in L^2_S[0,1]^n$ (uniquely determined by F)

the subset of $L^2(\Omega)$ generated by $I_n(f), f \in L^2_S[0,1]^n$ is called the Wiener chaos of order n

The Malliavin operators on Wiener chaos:
D is the Malliavin derivative and L is the Onstein-Uhlenbeck operator

$$D_s I_n(f) = n I_{n-1}(f(\cdot, s))$$

$$(-L)^{-1} I_n(f) = \frac{1}{n} I_n(f)(n \geq 1)$$

$$\delta I_n\left(f(\cdot, t)\right) = I_{n+1}(\tilde{f}).$$

Easy to see that

$$F = \delta D(-L)^{-1} F$$

if $F = I_n(f)$

Basic obervation: define

$$(\mathcal{N}f)(x,\mathbf{y}) = \sigma^2 \partial_x f(x,\mathbf{y}) - xf(x,\mathbf{y}),$$

for $f : \mathbb{R}^{d+1} \to \mathbb{R}$ differentiable.
Assume $X \sim N(0,\sigma^2)$ and X is independent of $\mathbb{Y}$. Then

$$\mathbf{E}\mathcal{N}f(X,\mathbb{Y}) = 0.$$

ideea of the proof: By the standard Stein method, for almost all $\mathbf{y} \in \mathbb{R}^d$,

$$\sigma^2 \mathbf{E} \partial_x f(X, \mathbf{y}) = \mathbf{E} X f(X, \mathbf{y})$$

or

$$\sigma^2 \int_{\mathbb{R}} \partial_x f(x, \mathbf{y}) dP_X(x) = \int_{\mathbb{R}} x f(x, \mathbf{y}) dP_X(x).$$

Let us integrate with respect to the probability measure $P_{\mathbb{Y}}$.

$$
\begin{aligned}
& \sigma^2 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} \partial_x f(x, \mathbf{y}) dP_X(x) \right) dP_{\mathbb{Y}}(\mathbf{y}) \\
= \;& \sigma^2 \int_{\mathbb{R}^{d+1}} \partial_x f(x, \mathbf{y}) dP_X(x) \otimes dP_{\mathbb{Y}}(\mathbf{y}) \\
= \;& \sigma^2 \int_{\mathbb{R}^{d+1}} \partial_x f(x, \mathbf{y}) dP_{(X,\mathbb{Y})}(x, \mathbf{y}) = \sigma^2 \mathbf{E} \partial_x f(X, \mathbb{Y}),
\end{aligned}
$$

where we used the independence of X and $\mathbb{Y}$ for the first equality on the above line.

Similarly,

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} x f(x, \mathbf{y}) dP_X(x) \right) dP_{\mathbb{Y}}(\mathbf{y}) \\ = & \int_{\mathbb{R}^{d+1}} x f(x, \mathbf{y}) dP_X(x) \otimes P_{\mathbb{Y}}(\mathbf{y}) = \int_{\mathbb{R}^{d+1}} x f(x, \mathbf{y}) dP_{(X, \mathbb{Y})}(x, \mathbf{y}) \\ = & \mathbf{E} X f(X, \mathbb{Y}). \end{aligned}$$

Conversely, let X be such that $\mathbf{E}|X| < \infty$. Assume

$$\mathbf{E}\mathcal{N}f(X, \mathbb{Y}) = 0$$

for any $f : \mathbb{R}^{d+1} \to \mathbb{R}$ such that $\|\partial_x f\|_\infty < \infty$. Then

$$X \sim N(0, \sigma^2)$$

and

$$X \text{ independent of } \mathbb{Y}.$$

Let φ be the characteristic function of the vector $(X, \mathbb{Y})$, i.e.

$$\varphi(\lambda_1, \boldsymbol{\lambda}) = \mathbf{E}\left(e^{i(\lambda_1 X + \boldsymbol{\lambda}\mathbb{Y})}\right),$$

for $\lambda_1 \in \mathbb{R}$ and $\boldsymbol{\lambda} \in \mathbb{R}^d$. By applying the one-dimensional Stein equation for the real and imaginary parts of φ, we get

$$\begin{aligned} \partial_{\lambda_1}\varphi(\lambda_1, \boldsymbol{\lambda}) &= i\mathbf{E}\left(Xe^{i(\lambda_1 X + \boldsymbol{\lambda}\mathbb{Y})}\right) \\ &= i\sigma^2\mathbf{E}\left(\partial_x e^{i(\lambda_1 X + \boldsymbol{\lambda}\mathbb{Y})}\right) = -\lambda_1\sigma^2\varphi(\lambda_1, \boldsymbol{\lambda}). \end{aligned}$$

By noticing that for every $\boldsymbol{\lambda} \in \mathbb{R}^d$, $\varphi(0, \boldsymbol{\lambda}) = \varphi_{\mathbb{Y}}(\boldsymbol{\lambda})$ (the characteristic function of the vector $\mathbb{Y}$), we obtain

$$\varphi(\lambda_1, \boldsymbol{\lambda}) = \varphi_{\mathbb{Y}}(\boldsymbol{\lambda}) e^{-\frac{\sigma^2 \lambda_1^2}{2}},$$

and this implies $X \sim N(0, \sigma^2)$ and X independent of $\mathbb{Y}$.

We associate the Stein's equation

$$(\mathcal{N} f)(x, \mathbf{y}) = h(x, \mathbf{y}) - \mathbf{E} h(Z, \mathbf{y}), \quad x \in \mathbb{R}, y \in \mathbb{R}^d$$

where $Z \sim N(0, \sigma^2)$ and $h : \mathbb{R}^{d+1} \to \mathbb{R}$ is bounded with bounded partial derivatives.
Then: the above equation admits a unique bounded solution f_h.
Moreover,

$$\|f_h\|_\infty \leq \|\partial_x h\|_\infty, \quad \|\partial_x f_h\|_\infty \leq C \|\partial_x h\|_\infty$$

and

$$\|\partial_{\mathbf{y}} f_h\|_\infty \leq C \|\partial_{\mathbf{y}} h\|_\infty.$$

Next, we follow the standard Stein-Malliavin route: Let

$$\theta = P_{(X,\mathbb{Y})} \text{ and } \eta = P_Z \otimes P_{\mathbb{Y}}.$$

Then

$$\begin{aligned} & \int_{\mathbb{R}^{d+1}} h(x,\mathbf{y})d\theta(x,\mathbf{y}) - \int_{\mathbb{R}^{d+1}} h(x,\mathbf{y})d\eta(x,\mathbf{y}) \\ = & \ \sigma^2 \mathbf{E}\partial_x f_h(X,\mathbb{Y}) - \mathbf{E}Xf_h(X,\mathbb{Y}) \end{aligned}$$

Write(if X is centered)

$$X = \delta D(-L)^{-1}X$$

and then, if $\mathbb{Y} = (Y_1, ..., Y_d)$,

$$\mathbf{E}Xf_h(X,\mathbb{Y}) = \mathbf{E}\langle D(-L)^{-1}X, Df_h(X,\mathbb{Y})\rangle$$
$$= \mathbf{E}\partial_x f_h(X,\mathbb{Y})\langle D(-L)^{-1}X, DX\rangle$$
$$+\mathbf{E}\sum_{j=1}^{d}\partial_{y_j} f_h(X,\mathbb{Y})\langle D(-L)^{-1}X, DY_j\rangle$$

Recall that f_h and its partial derivatives are bounded.

We obtain, if $X, Y_j \in \mathbb{D}^{1,2}$ for $j = 1, ..., d$, (by approximating the Lipschitz functions by differentiable functions with bounded partial derivatives)

$$\begin{aligned} & d_W(P_{(X,\mathbb{Y})}, P_Z \otimes P_{\mathbb{Y}}) \\ \leq \quad & C \left[\mathbf{E} \left| \sigma^2 - \langle DX, D(-L)^{-1} X \rangle \right| + \right. \\ & \left. + \sum_{j=1}^{d} \mathbf{E} \left| \langle D(-L)^{-1} X, DY_j \rangle \right| \right]. \end{aligned}$$

Wiener chaos

Assume $(X_k, k \geq 1)$ is a sequence in the p Wiener chaos,

$$X_k = I_p(f_k).$$

Assume

$$X_k \to^{(d)}_{k\to\infty} Z \sim N(0, \sigma^2).$$

Let $(\mathbb{Y}_k, k \geq 1) = (Y_{1,k}, ..., Y_{d,k})$ be a d-dimensional random sequence such that

$$\mathbb{Y}_k \to \mathbb{U} \text{ in some sense,}$$

with $\mathbb{U}$ an arbitrary random vector.

The purpose is to discuss the joint convergence of

$$(X_k, \mathbb{Y}_k)$$

to

$$(Z, \mathbb{U}),$$

with $Z, \mathbb{U}$ independent.

The celebrated Fourth Moment Theorem

Theorem

Fix an integer $n \geq 1$. Consider a sequence $(F_k = I_n(f_k), k \geq 1)$ of square integrable random variables in the nth Wiener chaos. Assume that

$$\lim_{k\to\infty} \mathbf{E}[F_k^2] = \lim_{k\to\infty} n!\|f_k\|^2_{H^{\odot n}} = 1.$$

Then, the following statements are equivalent.

1. $(F_k = I_n(f_k), k \geq 1) \to^{(d)} Z \sim N(0,1)$
2. $\lim_{k\to\infty} \mathbf{E}[F_k^4] = 3.$
3. $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{H^{\otimes 2(n-l)}} = 0$ *for* $l = 1, 2, \ldots, n-1.$
4. $\|DF_k\|^2_H$ *converges to* n *in* $L^2(\Omega)$ *as* $k \to \infty$.

A key lemma

Lemma

Let $p \geq 2$ and $q \geq 1$ be two integer numbers. Let $(X_k, k \geq 1)$ be that such for every $k \geq 1$, $X_k = I_p(f_k)$ with $f_k \in H^{\odot p}$. Assume

$$X_k \to^{(d)}_{k\to\infty} Z \sim N(0, \sigma^2).$$

Then, for every $g \in H^{\odot q}$,

$$\|f_k \otimes_r g\|_{H^{p+q-2r}} \to_{k\to\infty} 0 \text{ for every } \begin{cases} r = 1, ..., p \wedge q \text{ if } p \neq q \\ r = 1, ..., (p \wedge q) - 1 \text{ if } p = q. \end{cases}$$

The result is valid if we replace g by $(g_k, k \geq 1)$ convergent in $H^{\odot q}$.

Recall: if $\mathcal{H} = L^2(T)$, the contraction $f \otimes_r g$ is the function in $L^2(T^{p+q-2r})$ given by, for $r = 1, ..., n$,

$$(f \otimes_r g)(t_1, ..., t_{p+q-2r})$$
$$= \int_{T^r} f(u_1, ..., u_r, t_1, ..., t_{p-r}) g(u_1, ..., u_r, t_{p-r+1}, ..., t_{p+q-2r})$$
$$du_1 ... du_r$$

and $f \otimes_0 g = f \otimes g$, the tensor product.

General assumptions:

1.
$$X_k \rightarrow^{(d)}_{k\to\infty} Z \sim N(0,\sigma^2). \tag{1}$$

2.
$$\mathbb{Y}_k \rightarrow_{k\to\infty} \mathbb{U} \text{ in } L^2(\Omega). \tag{2}$$

   where $\mathbb{U}$ is an arbitrary random vector.

3. $X_k, Y_{j,k}$ are asymptotically uncorrelated, i.e. for every $j = 1, ..., d$.

$$\mathbf{E} X_k Y_{j,k} \rightarrow_{k\to\infty} 0 \tag{3}$$

Fixed Wiener chaos

Let $p \geq 2$ and let $q_1, ..., .q_d \geq 1$ be integer numbers. Assume that $(X_k, k \geq 1)$ is such that $X_k = I_p(f_k), f_k \in H^{\odot p}$ and (1) holds true.

Let $(\mathbb{Y}_k, k \geq 1) = ((Y_{1,k}, ..., Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that for every $k \geq 1, j = 1, ..., d$,

$$Y_{j,k} = I_{q_j}(g_{j,k}) \text{ with } g_{j,k} \in \mathcal{H}^{\odot q_j}.$$

Suppose (2) and (3). Then

$$(X_k, \mathbb{Y}_k) \to_{k\to\infty}^{(d)} (Z', \mathbb{U}),$$

where $Z' \sim N(0, \sigma^2)$ and Z' is independent by $\mathbb{U}$.

Moreover, for k sufficiently large,

$$d_W\left(P_{(X_k,Y_k)}, P_{Z'} \otimes P_{\mathbb{U}}\right) \leq C\left[\mathbf{E}\left|\sigma^2 - \langle D(-L)^{-1}X_k, DX_k\rangle\right| + \sum_{j=1}^{d} \mathbf{E}\left|\langle D(-L)^{-1}X_k, DY_{j,k}\rangle_H\right|\right] + d_W(\mathbb{Y}_k, \mathbb{U}).$$

Proof

First, we have the Stein-Malliavin bound from the general theory and the triangle inequality. Indeed, for every $k \geq 1$,

$$\begin{aligned} & d_W\left(P_{(X_k,\mathbb{Y}_k)}, P_{Z'} \otimes P_{\mathbb{U}}\right) \\ \leq\ & d_W\left(P_{(X_k,\mathbb{Y}_k)}, P_{Z'} \otimes P_{\mathbb{Y}_k}\right) + d_W\left(P_{Z'} \otimes P_{\mathbb{Y}_k}, P_{Z'} \otimes P_{\mathbb{U}}\right) \\ \leq\ & d_W\left(P_{(X_k,\mathbb{Y}_k)}, P_{Z'} \otimes P_{\mathbb{Y}_k}\right) + d_W\left(P_{\mathbb{Y}_k}, P_{\mathbb{U}}\right) \\ =\ & d_W(\theta_k, \eta_k) + d_W\left(P_{\mathbb{Y}_k}, P_{\mathbb{U}}\right) \end{aligned}$$

where

$$\theta_k = P_{(X_k,\mathbb{Y}_k)}, \;\; \eta_k = P_Z \otimes P_{\mathbb{Y}_k}, \;\; \eta = P_Z \otimes P_{\mathbb{U}}.$$

In this step, we prove that

$$d_W(\theta_k, \eta_k) \to_{k\to\infty} 0.$$

From the Stein-Malliavin bound that

$$\begin{aligned} d_W(\theta_k, \eta_k) \quad \leq \quad & C \Bigg[\left(\mathbf{E} \left(\langle D(-L)^{-1} X_k, DX_k \rangle - \sigma^2 \right)^2 \right)^{\frac{1}{2}} \\ & + \sum_{j=1}^{d} \left(\mathbf{E} \langle D(-L)^{-1} X_k, DY_{j,k} \rangle^2 \right)^{\frac{1}{2}} \Bigg] \end{aligned}$$

The assumption (1) and the Fourth Moment Theorem implies that

$$\mathbf{E} \left(\langle D(-L)^{-1} X_k, DX_k \rangle - \sigma^2 \right)^2 \to_{k\to\infty} 0.$$

for every $k \geq 1$ and $j = 1, ..., d$,

$$\mathbf{E}\langle D(-L)^{-1}X_k, DY_{j,k}\rangle^2 = (\mathbf{E}X_k Y_{j,k})^2 1_{p=q_j} + \sum_{r=1}^{p\wedge q_j} c(r,p,q)\|f_k \widetilde{\otimes}_r g_{j,k}\|^2_{H^{\otimes p+q_j-2r}},$$

where $c(r,p,q_j)$ are constants and $c(p\wedge q_j, p, q) = 0$ if $p \neq q_j$. By the key lemma,

$$\|f_k \widetilde{\otimes}_r g_{j,k}\|^2_{H^{\otimes p+q_j-2r}} \to_{k\to\infty} 0$$

for every $r = 1, ..., p \wedge q_j$ (if $p \neq q_j$) and $r = 1, ..., (p \wedge q_j) - 1$ (if $p = q_j$).

Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be a continuous and bounded function. By using the triangle's inequality, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{d+1}} f(x) d\theta_k(x) - \int_{\mathbb{R}^{d+1}} f(x) d\eta(x) \right| \\
&\leq \left| \int_{\mathbb{R}^{d+1}} f(x) d\theta_k(x) - \int_{\mathbb{R}^{d+1}} f(x) d\eta_k(x) \right| \\
&+ \left| \int_{\mathbb{R}^{d+1}} f(x) d\eta_k(x) - \int_{\mathbb{R}^{d+1}} f(x) d\eta(x) \right| .
\end{aligned}$$

The first summand in the right-hand side converges to zero as $k \to \infty$ by Step 2. The second summand in the right-hand side also goes to zero as k tends to infinity due to the assumption (2). Then, the conclusion is obtained. □

Finite sum of Wiener chaoses

Assume $(X_k, k \geq 1)$ is as before and let

$$\mathbb{Y}_k = (Y_{1,k}, ..., Y_{d,k})$$

be such that for every $j = 1, ..., d$ and for every $k \geq 1$,

$$Y_{j,k} = \sum_{n=0}^{N_0} I_n(g_{n,k}^{(j)}),$$

with $N_0 \geq 1$, $g_{n,k}^{(j)} \in H^{\odot n}$ for $n \geq 0, k \geq 1$ and $= 1, ..., d$.

Assume (2) and (3). Then

$$(X_k, \mathbb{Y}_k) \to^{(d)}_{k\to\infty} (Z', \mathbb{U})$$

where $Z \sim N(0, \sigma^2)$ and $Z', \mathbb{U}$ are independent. Moreover, the estimate Stein-Malliavin estimate holds true.

General result

Theorem

Let us consider the integrer numbers $p \geq 2$, $d \geq 1$. Let $(X_k, k \geq 1)$ be a sequence of random variables such that for every $k \geq 1$, $X_k = I_p(f_k)$ with $f_k \in H^{\odot p}$. Assume that

$$X_k \to^{(d)}_{k \to \infty} Z \sim N(0, \sigma^2).$$

Let $(\mathbb{Y}_k, k \geq 1) = ((Y_{1,k}, ..., Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that, for every $j = 1, ..., d$, the random variable $Y_{j,k}$ belongs to $\mathbb{D}^{1,4}$

Theorem

(**Continuation**) *and it admits the chaos expansion*

$$Y_{j,k} = \sum_{n=0}^{\infty} I_n(g_{n,k}^{(j)}) \text{ with } g_{n,k}^{(j)} \in H^{\odot n}$$

and

$$\sup_{k\geq 1} \sum_{n=M+1}^{\infty} n! \|g_{n,k}\|_{H^{\otimes n}}^{2} \to_{M\to\infty} 0. \tag{4}$$

Theorem

(Continuation) *Suppose that there exists a random vector $\mathbb{U}$ in $\mathbb{R}^d$ such that*

$$\mathbb{Y}_k \to^{(d)}_{k\to\infty} \mathbb{U}.$$

Then, if

$$\mathbf{E} X_k Y_{j,k} \to_{k\to\infty} 0 \text{ for every } j = 1, \ldots, d$$

we have

$$(X_k, \mathbb{Y}_k) \to_{k\to\infty} (Z', \mathbb{U}) \text{ in law.}$$

where $Z' \sim N(0, \sigma^2)$ and Z' is independent by the random vector $\mathbb{U}$.

Moreover, for k sufficiently large,

$$
\begin{aligned}
& d_W\left(P_{(X_k,Y_k)}, P_{Z'} \otimes P_{\mathbb{U}}\right) \\
& \leq C\left[\mathbf{E}\left|\sigma^2 - \langle D(-L)^{-1}X_k, DX_k\rangle\right|\right. \\
& \left.+ \sum_{j=1}^{d} \mathbf{E}\left|\langle D(-L)^{-1}X_k, DY_{j,k}\rangle_H\right|\right] + d_W(\mathbb{Y}_k, \mathbb{U}).
\end{aligned}
$$

- Condition (4) is automatically verified when
  1. $X_{j,k}$ belongs to a finite sum of Wiener chaoses
  2. or when $Y_{j,k} = Y_j$ for every $k \geq 1$.
- The assumption (4) is not new, it also appears in the context of the normal approximation of Wiener space

The uncorrelation condition (3) is obviously crucial for the joint convergence of $(X_k, \mathbb{Y}_k)$.

Another interesting question is what happens if we assume, instead of (3), that

$$\mathbf{E} X_k Y_{j,k} \to_{k\to\infty} c_j,$$

with $c_j \neq 0$ for $j = 1, ..., d$. Can we deduce the joint convergence of $(X_k, \mathbb{Y}_k)$ to a random vector with marginals Z and $\mathbb{U}$?

In the case when $\mathbb{U}$ follows a Gaussian distribution, the answer is given by the multidimensional Fourth Moment Theorem.

In order to give a complete answer, we need to know how to characterize the law of the vector $(Z, \mathbb{U})$ when $Z \sim N(0, \sigma^2)$ is not independent of $\mathbb{U}$ and the law of $\mathbb{U}$ is not Gaussian.

For $M \geq 1$, let us define,

$$Y_{j,k}^{M} = \sum_{n=0}^{M} I_n(g_{n,k}^{(j)}),$$

and consider the random vector in $\mathbb{R}^d$

$$\mathbb{Y}_k^M = (Y_{1,k}^M, ..., Y_{d,k}^M), \quad k \geq 1.$$

Clearly, for every $k \geq 1$,

$$\mathbf{E}\|\mathbb{Y}_k^M - \mathbb{Y}_k\|_{\mathbb{R}^d}^2 \to_{M\to\infty} 0.$$

By (3) and the orthogonality of multiple stochastic integrals of different orders, for every $j = 1, ..., d$ and for every $M \geq 1$,

$$\mathbf{E}X_k Y_{j,k}^M \to_{k\to\infty} 0.$$

$$
\begin{aligned}
& \left| \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{U} \rangle_{\mathbb{R}^d}} \right| \\
\leq \quad & \left| \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} \right| \\
& + \left| \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} \right| \\
& + \left| \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{U} \rangle_{\mathbb{R}^d}} \right| \\
= \quad & a_{M,k} + b_{M,k} + c_{M,k}.
\end{aligned}
$$

Estimation of $a_{M,k}$.

By the mean value theorem,

$$
\begin{aligned}
a_{M,k} &\leq \mathbf{E}\left| e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k\rangle_{\mathbb{R}^d}} - e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M\rangle_{\mathbb{R}^d}} \right| \leq \mathbf{E}\|\mathbb{Y}_k^M - \mathbb{Y}_k\|_{\mathbb{R}^d} \\
&\leq \sqrt{\sum_{j=1}^{d} \mathbf{E}\left(Y_{j,k}^M - Y_{j,k}\right)^2} = \sqrt{\sum_{j=1}^{d} \sum_{n=M+1}^{\infty} n! \|g_{n,k}^{(j)}\|^2_{H^{\otimes n}}} \\
&\leq \sqrt{\sum_{j=1}^{d} \sup_{k\geq 1} \sum_{n=M+1}^{\infty} n! \|g_{n,k}^{(j)}\|^2_{H^{\otimes n}}}
\end{aligned}
$$

and the last quantity goes to zero as $M \to \infty$

$b_{M,k}$ goes to zero as $k \to \infty$ because $\mathbb{Y}^M$ belongs to a finite sum of Wiener chaoses.

$c_{M,k}$ goes to zero because $\mathbb{Y}_k^M$ converges to $\mathbb{Y}_k$ in $L^2(\Omega)$ as $M \to \infty$ and $\mathbb{Y}_k$ converges to $\mathbb{U}$ in law as $k \to \infty$.

It is possible to assume only the convergence in law of the sequence $(\mathbb{Y}_k, k \geq 1)$ instead of (2) if the components of $\mathbb{Y}_k$ belongs to the sum of the first q Wiener chaos with $q \leq p$.

Let us consider the integer numbers $p \geq 2$, $d \geq 1$. Let $(X_k, k \geq 1)$ be a sequence of random variables such that for every $k \geq 1$, $X_k = I_p(f_k)$ with $f_k \in H^{\odot p}$ that satisfies (1).
Let $(\mathbb{Y}_k, k \geq 1) = ((Y_{1,k}, ..., Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that, for every $j = 1, ..., d$, the random variable $Y_{j,k}$ belongs to $\mathbb{D}^{1,2}$, and it admits the chaos expansion

$$Y_{j,k} = \sum_{n=0}^{q} I_n(g_{n,k}^{(j)}) \text{ with } g_{n,k}^{(j)} \in H^{\odot n}$$

with $q \leq p$.

Suppose that there exists a random vector $\mathbb{U}$ in $\mathbb{R}^d$ such that

$$\mathbb{Y}_k \to^{(d)}_{k\to\infty} \mathbb{U}. \tag{5}$$

Then, if (3) holds true, we have

$$(X_k, \mathbb{Y}_k) \to^{(d)}_{k\to\infty} (Z', \mathbb{U}),$$

where $Z' \sim N(0, \sigma^2)$ and Z' is independent by the random vector $\mathbb{U}$. Moreover, (5) holds true.

A Central-Noncentral Limit Theorem

Let $(B^H, t \geq 0)$ be a fractional Brownian motion with Hurst index $H \in (0,1)$. For $N \geq 1$, define

$$V_N = q! \frac{1}{\sqrt{N}} \sum_{k=0}^{N} H_q \left(B^H_{k+1} - B^H_k \right),$$

where H_q is the Hermite polynomial of degree q. Then, the Breuer-Major theorem states that, if $H \in \left(0, 1 - \frac{1}{2q}\right)$ the sequence $(V_N, N \geq 1)$ converges to a Gaussian random variable $Z \sim N(0, \sigma^2_{q,H})$, where the variance $\sigma^2_{q,H}$ is explicily known.

On the other hand, the sequence $(U_N, N \geq 1)$ given by

$$U_N = 2N^{1-2H} \sum_{i=0}^{N-1} H_2\left(B_{k+1}^H - B_k^H\right), \qquad N \geq 1,$$

converges in distribution, for $H \in \left(\frac{3}{4}, 1\right)$, to $c_{2,H} R^{(2H-1)}$ where $R^{(2H-1)}$ is a Rosenblatt random variable with Hurst parameter $2H-1$ and again the constant $c_{2,H} > 0$ is known.

We can show the random sequence (V_N, U_N) converges in law, as $N \to \infty$, to

$$(Z, c_{2,H} R^{(2H-1)})$$

with Z independent of $R^{(2H-1)}$. This can be obtained from some findings in the literature but it also follows from our results.

The purpose is to find the rate of convergence, under the Wasserstein distance, for this two-dimensional limit theorem.

Let us assume in the sequel

$$H \in \left(\frac{3}{4}, 1 - \frac{1}{2q}\right) \text{ and } q \geq 3.$$

By using the Stein-Malliavin bound, we have

$$\begin{aligned} & d_W\left(P_{(V_N, U_N)}, P_Z \otimes P_{c_{2,H} R^{(2H-1)}}\right) \\ \leq \quad & C\left[\left(\mathbf{E}\left(\sigma^2 - \langle DV_N, D(-L)^{-1} V_N\rangle\right)^2\right)^{\frac{1}{2}} + d_W(P_{U_N}, P_{c_{2,H} R^{(2H-1)}}) \right. \\ & \left. + \sqrt{\mathbf{E}\left(\langle DV_N, DU_N\rangle\right)^2}\right]. \end{aligned}$$

We known the rate of convergence to their limits for each of the sequences $(V_N, N \geq 1)$ and $(U_N, N \geq 1)$.

$$\left(\mathbf{E}\left(\sigma^2 - \langle DV_N, D(-L)^{-1}V_N\rangle\right)^2\right)^{\frac{1}{2}}$$
$$\leq C_{H,q}\begin{cases} N^{H-1} \text{ if } H \in \left(\frac{3}{4}, \frac{2q-3}{2q-2}\right] \\ N^{qH-q+\frac{1}{2}} \text{ if } H \in \left(\frac{2q-3}{2q-2}, \frac{2q-1}{2q}\right) \end{cases}.$$

Moreover,

$$d_W(U_N, c_{2,H}R^{(2H-1)}) \leq C_H N^{\frac{3}{2}-2H}.$$

We estimate the quantity $\sqrt{\mathbf{E}\left(\langle DV_N, DU_N\rangle\right)^2}$. We have

$$\begin{aligned}
&\mathbf{E}\langle DV_N, DU_N\rangle^2 \\
\leq\ & c_q N^{1-4H}\left[\sum_{i,j,k,l=1}^{N}\langle h_i, h_k\rangle^{q-1}\langle h_i, h_j\rangle\langle h_k, h_l\rangle\langle h_j, h_l\rangle\right. \\
&+\sum_{i,j,k,l=1}^{N}\langle h_i, h_k\rangle^{q-2}\langle h_i, h_j\rangle\langle h_k, h_l\rangle\langle h_i, h_l\rangle\langle h_j, h_k\rangle \\
&\left.+\sum_{i,j,k,l=1}^{N}\langle h_i, h_k\rangle^{q-2}\langle h_i, h_j\rangle^2\langle h_k, h_l\rangle^2\right] =: a_{1,N}+a_{2,N}+a_{3,N}.
\end{aligned}$$

The first summand can be decomposed into a sum of 15 terms....

We found

$$\mathbf{E}\langle DV_N, DU_N\rangle^2 \leq c_{q,H} \begin{cases} N^{2H-2} \text{ if } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)}\right) \\ N^{(2H-2)q+1} \text{ if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right), \end{cases} ,$$

the bound on the first branch being immaterial for $q = 3, 4$.

Finally

$$d_W\left((V_N, U_N), (Z, c_{2,H}R^{(2H-1)})\right)$$
$$\leq c_{q,H}\begin{cases} N^{H-1} + N^{\frac{3}{2}-2H} \text{ for } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)}\right) \\ N^{(H-1)q+\frac{1}{2}} + N^{\frac{3}{2}-2H} \text{ for } \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right) \end{cases}$$

Infinite chaos expansion

Let $(W(h), h \in H)$ be an isonormal process and let $(h_i, i \geq 1)$ be a family of elements of H such that for every $i, j \geq 1$

$$\langle h_i, h_j \rangle_H = \rho_H(i-j),$$

where ρ_H is the auto-correlation function of the fractional noise. Consider the sequence $(V_N, N \geq 1)$ given by

$$V_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} I_p(h_k^{\otimes p}).$$

8and let

$$Y = e^{W(h_1)} = \sum_{n \geq 0} \frac{1}{n!} I_n(h_1^{\otimes n}).$$

Assume

$$0 < H < 1 - \frac{1}{2p}.$$

We know that $(V_N, N \geq 1)$ converges in law, as $N \to \infty$, to $Z \sim N(0, \sigma^2_{p,H})$. Moreover, we have the following estimate for the Wasserstein distance: if N is large,

$$d_W(V_N, Z) \leq C \begin{cases} n^{-\frac{1}{2}}, \text{ if } H \in (0, \frac{1}{2}] \\ n^{H-1}, \text{ if } H \in [\frac{1}{2}, \frac{2p-3}{2p-2}) \\ n^{pH-p+\frac{1}{2}}, \text{ if } H \in [\frac{2p-3}{2p-2}, \frac{2p-1}{2p}). \end{cases}$$

We check the joint convergence in law of the couple (X_N, Y) when $N \to \infty$ and we evaluate the Wasserstein distance associated to it. we get for N large

$$d_W(P_{(V_N,Y)}, P_Z \otimes P_Y) \leq C \begin{cases} n^{-\frac{1}{2}}, \text{ if } H \in (0, \frac{1}{2}] \\ n^{H-1}, \text{ if } H \in [\frac{1}{2}, \frac{3}{4}) \\ n^{H-1} + n^{pH-p+\frac{1}{2}}, \text{ if } H \in [\frac{3}{4}, \frac{2p-1}{2p}). \end{cases}$$