Increasing paths in random temporal graphs

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based on joint work with
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temporal graphs

A temporal graph $G = (V, E, \pi)$ is a finite simple graph together with an ordering $\pi : E \to \{1, 2, \ldots, |E|\}$ of the edges.

The edges have time stamps.

Edge e precedes edge f if $\pi(e) < \pi(f)$.

Temporal graphs model time-dependent propagation processes such as infection processes.

An infection spreads along monotone increasing paths.

random simple temporal graphs

Today $G \sim \mathcal{G}(n, p)$ is an Erdős–Rényi random graph.

Such a random graph may be generated by assigning i.i.d. random labels to each edge of the complete graph K_n.

The label of edge $\{i, j\}$ is an exponential random variable $W_{i,j}$.

An edge is kept if and only if $W_{i,j} \le -\log(1-p)$ (so that $\mathbb{P}\{W_{i,j} \le \tau\} = p$).

Denote the random temporal graph by $G(W)$.

connectivity

This model was formally introduced by Casteigts, Raskin, Renken, and Zamaraev (2022) and Becker, Casteigts, Crescenzi, Kodric, Renken, Raskin, and Zamaraev (2022).
They prove that, with high probability,

- a typical pair of vertices is connected by an increasing path if $p \geq (1+\epsilon)\log n/n$ and disconnected if $p \leq (1-\epsilon)\log n/n$.
- a typical vertex can reach all other vertices if $p \geq (2+\epsilon)\log n/n$ and cannot reach all of them if $p \leq (2-\epsilon)\log n/n$.
- any pair of vertices are connected if $p \geq (3+\epsilon)\log n/n$ but not all of them are connected if $p \leq (3-\epsilon)\log n/n$.

longest and shortest monotone paths

Define $\ell(i,j)$ and $L(i,j)$ as the minimum and maximum length of any increasing path from i to j.

We study

- $L(1,2)$ and $\ell(1,2)$; the lengths of the longest and shortest increasing paths between two fixed vertices.
- $\max_{j\in\{2,\ldots,n\}} L(1,j)$ and $\max_{j\in\{2,\ldots,n\}} \ell(1,j)$; the maximum length of the longest and shortest increasing paths starting at a fixed vertex;
- $\max_{i,j\in[n]} L(i,j)$ and $\max_{i,j\in[n]} \ell(i,j)$; the maximal length of the longest and shortest increasing paths.

longest monotone paths

Angel, Ferber, Sudakov, and Tassion (2020) showed that if $p = o(n)$ and, $pn/\log n \to \infty$ then, with high probability,

$$\max_{i,j \in [n]} L(i,j) \sim enp\ .$$

This excludes the “interesting” regime $p \sim c \log n/n$.

In fact, when $pn/\log n \to \infty$,

$$L(1,2) \sim enp\ .$$

proof of $L(1,2) \sim enp$

Partition $[0, p)$ as $[0, 2\log n/n) \cup [2\log n/n, p - 2\log n/n) \cup [p - 2\log n/n, p)$ and let W_1, W_2, W_3 be the collections of edge weights falling in the corresponding intervals.

This decomposes $G(W)$ into the union of three graphs $G(W_1), G(W_2), G(W_3)$.

The longest monotone path in $G(W_2)$ has length $\sim enp$, say from i^* to j^*.

But in $G(W_1)$ there is a path from vertex 1 to i^* and in $G(W_3)$ there is a path from vertex j^* to 2.

some key constants

For any $c > 0$, define

$$\alpha(c) = \inf\{x > 0 : x \log(x/ec) = 1\}$$

and for $c > 1$,

$$\beta(c) = \sup\{x > 0 : x \log(x/ec) = -1\}$$

$$\gamma(c) = \inf\{x > 0 : x \log(x/ec) = -1\}$$

some key constants

The equation $x \log(x/ec) = -1$ has at most two solutions for $c > 0$, these are $\beta(c)$ and $\gamma(c)$.

When $c = 1$, there is only one solution and $\beta(1) = \gamma(1) = 1$.

For $c < 1$ there is no solution, for $c > 1$ there are two.

The equation $x \log(x/ec) = 1$ has a unique solution for all $c > 0$.

Note that as $c \to \infty$, $\alpha(c)/c \to e$, $\beta(c)/c \to e$, and $\gamma(c)/c \to 0$.

An example value is $\alpha(1) \approx 3.5911$.

longest monotone paths

Suppose $p = c \log n/n$. With high probability,

- if $c \in (0, 1)$, there is no increasing path between 1 and 2, and if $c \geq 1$, $L(1, 2) \sim \beta(c) \log n$;
- for all $c > 0$, $\max_{j \in \{2,\dots,n\}} L(1, j) \sim ec \log n$;
- for all $c > 0$, $\max_{i,j \in [n]} L(i, j) \sim \alpha(c) \log n$.

shortest monotone paths

Let $p = c \log n / n$. With high probability,

- for $c > 1$, $\ell(1, 2) \sim \gamma(c) \log n$;
- for $c > 2$, $\max_{i \in [n]} \ell(1, i) \sim \gamma(c - 1) \log n$;
- for $c > 3$, $\max_{i,j \in [n]} \ell(i, j) \sim \gamma(c - 2) \log n$.

upper bounds: first moment considerations

Let X_k be the number of increasing paths of length k. Then

$$\mathbb{E}X_k = \binom{n}{k+1}(k+1)!\frac{p^k}{k!} \sim n\left(\frac{nep}{k}\right)^k$$

Similarly, for the number Y_k of increasing paths of length k starting at vertex 1 and for the number Z_k of increasing paths of length k vertex 1 to vertex 2,

$$\mathbb{E}Y_k = \binom{n-1}{k}k!\frac{p^k}{k!} \sim \left(\frac{nep}{k}\right)^k$$

$$\mathbb{E}Z_k = \binom{n-2}{k-1}(k-1)!\frac{p^k}{k!} \sim \frac{1}{n}\left(\frac{nep}{k}\right)^k$$

The upper bounds follow simply from these identities.

We prove the lower bounds of the three statements by three different techniques.

a related model: fitst-passage percolation

Equip the complete graph K_n with inependent exponential weights on every edge.

The weight of a path is the sum of the weights on the edges of the path.

Janson (1999) noted that the length of the minimum weight path between two typical vertices is $\sim \log n$ and the maximum length of any shortest path starting from vertex 1 is $\sim e \log n$.

Addario-Berry, Broutin, and Lugosi (2010) prove that the maximum length of all minimum weight paths is $\sim \alpha(1) \log n$ where $\alpha(1) \approx 3.5911$. This corresponds to the maximum height of n independent URRT's.

Observe that the shortest-path tree rooted at any vertex is a uniform random recursive tree.

proof of $\max_j L(1,j) \geq ec(1-o(1))\log n$

Since the bound is linear in c, it suffices to prove the lower bound for $c \leq 1$. Otherwise we may decompose the graph into $\lceil c \rceil$ disjoint layers and concatenate the paths.

When $c \leq 1$, one can show that the graph contains a uniform random recursive tree of size $n^{c(1-o(1))}$, rooted at vertex 1 such that all paths of the tree starting at vertex 1 are monotone.

The construction of the tree is similar to the shortest-path tree. We need to discard a small number of vertices in order to keep monotonicity.

Since the height of the URRT is $\sim e \log n^{c(1-o(1))}$, we have a monotone path of desired length.

This also shows that at least $n^{c(1-o(1))}$ vertices can be reached from vertex 1.

proof of $\max_{i,j} L(i,j) \geq \alpha(c)(1 - o(1)) \log n$

Since the expected number of monotone paths of length $\alpha(c)(1 - o(1)) \log n$ goes to infinity, it is natural to resort to the second moment method.

However, the second moment is too large due to the many ways paths can intersect.

We borrow ideas from Addario-Berry, Broutin, and Lugosi (2010) and apply the second moment method to a restricted class of paths.

proof of $\max_{i,j} L(i,j) \geq \alpha(c)(1-o(1)) \log n$

For a path P of length $k \sim \alpha(c) \log n$, the edge labels on it increase significantly more slowly than on a typical path.

So if we enforce the property that all the vertices on P are *typical*, so that the paths leaving them increase and decrease at the rate at most $1/(en)$, then all the paths leaving P are shorter than the corresponding segment of P.

This avoids undesired intersections and the second moment method works.

One needs to show that restricting the collection of paths does not significantly decrease their expected number.

building short and long monotone paths

Suppose $c > 1$. It still remains to show that

$$\ell(1,2) \leq (\gamma(c) + o(1)) \log n$$

and

$$L(1,2) \geq (\beta(c) - o(1)) \log n$$

Recall that $\gamma(c) < \beta(c)$ are the two solutions of
$x \log(x/ec) = -1$.
We prove that for any $x \in (\gamma(c), \beta(c))$, whp there exists an increasing path between 1 and 2 containing $\sim x \log n$ edges.

Note that $\gamma(1) = \beta(1) = 1$ so for $c \approx 1$, all monotone paths between 1 and 2 have about the same length $\log n$.

building short and long monotone paths

It is convenient to work with uniform $[0, 1]$ edge weights.

We look for increasing paths from vertex 1 such that labels increase as they should to have length $x \log n$.

Similarly, we look for decreasing paths from vertex 2

We conduct this search up to distance $\frac{1}{2} x \log n$.

We show that the two sets of end points of the path must intersect, because the sets at distance $\frac{x}{2} \log n$ are of size at least $n^{1/2}$.

building short and long monotone paths

We partition the interval $[0, p/2]$ into $r = (\log n)/(2A)$ disjoint intervals I_j of length cA/n where A is an appropriate constant.

We construct a supercritical branching process of monotone paths starting at vertex 1. At each level j, only edges are chosen with labels in the j-th interval.

The first interval is used to ensure that we have enough starting points.

Each path has $1 + (r-1)xA \approx \frac{1}{2}x \log n$ edges.

building short and long monotone paths

For a given vertex u, let $S_j(u)$ be the set of vertices v such that the graph $G(I_j)$ contains an increasing path of length xA from u to v. Then

$$\mathbb{E}|S_j(u)| \sim \binom{n}{xA} \left(\frac{cA}{n}\right)^{xA}$$

When $x \in (\gamma(c), \beta(c))$, this is greater than 1 for a well chosen A.

Possible collisions need to be taken care of.

the temporal diameter

The last argument is to show why

- for $c > 2$, $\max_{i \in [n]} \ell(1, i) \sim \gamma(c-1) \log n$;
- for $c > 3$, $\max_{i,j \in [n]} \ell(i, j) \sim \gamma(c-2) \log n$.

For $c > 2$, if we partition $(0, c \log n / n) = I_1 \cup I_2$ with I_2 of length $(1-\epsilon) \log n / n$, then $G(I_2)$ has an isolated vertex and in $G(I_1)$ shortest paths are of length at least $\gamma(c-1) \log n$.

temporal cliques

A set C of vertices is a temporal clique if for any $i, j \in C$, there is a monotone path from i to j in $G(W)$.

What is the typical size $\omega_{n,p}$ of the largest temporal clique?

As mentioned before Casteigts, Raskin, Renken, and Zamaraev (2022) showed that

$$\text{if } p \geq (3+\epsilon)\log n/n, \text{ then } \omega_{n,p} = n,$$

with high probability.

Becker, Casteigts, Crescenzi, Kodric, Renken, Raskin, and Zamaraev (2023) prove that

$$\text{if } p \leq (1-\epsilon)\log n/n, \text{ then } \omega_{n,p} = o(n),$$

while

$$\text{if } p \geq (1+\epsilon)\log n/n, \text{ then } \omega_{n,p} = n - o(n).$$

temporal cliques – subcritical case

When $p = c \log n/n$ for $c < 1$, we can say much more. The largest clique is of constant(!) size. Whp,

$$\omega_{n,p} \leq \left\lceil \frac{1}{1-c} + 1 \right\rceil$$

(Atamanchuk, Devroye, Lugosi (2024)).

For $c \leq 1/2$ the upper bound equals 3 which is best possible since any triangle in $\mathcal{G}(n,p)$ is a temporal clique.

We conjecture that the bound is sharp for all $c < 1$.

The proof is based on estimates of (the moments of) the size of temporal branching processes.

some further questions

- Length of longest path when p is constant?
- Size of largest temporal clique in the critical regime $p \sim \log n/n$?
- Different random graph models.
- Models with recovery, reinfection.
- Super spreader events.
- Statistical questions.