Diagonal Ramsey numbers

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based on joint work with:

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What is Ramsey Theory?

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The study of inevitable structure....

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Schur (1912) $x+y=z$	Rado (1930) $A\underset{\sim}{x}=\underset{\sim}{0}$	Hindman (1971) Finite sums $FS(x_1,x_2,\dots)$
Van der Waerden ('27) $a, a+d, a+2d, \dots,$	Hales-Jewett (1963) subspaces	• Euclidean Ramsey Theory • Inc./Dec. sequences • Canonical Ramsey

Who was Ramsey?

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<u>Theorem</u> (1930) In every red-blue colouring of pairs of integers there exists an <u>infinite</u> set $S \subseteq \mathbb{Z}$ such that the pairs in S are <u>monochromatic</u>.

Def $R(k,\ell) = \min\{n :$ every red-blue colouring of $E(K_n)$ contains K_k or $K_\ell\}$

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Improvements on UB

Thomason ('88): WINS $\sqrt{k}$

Conlon ('09): WINS $e^{c(\log k)^2/\log\log k}$

Sah ('20): WINS $e^{c(\log k)^2}$

Theorem $\exists \varepsilon > 0$

$$R(k,k) < (4-\varepsilon)^k$$

Off-diagonal

Erdős-Szekeres

$$R(k,\ell) \le \binom{k+\ell}{\ell} \approx \gamma^{-\ell}(1-\gamma)^{-k}$$

where $\gamma = \dfrac{\ell}{k+\ell}$

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Theorem 2: If $\ell \leq \frac{k}{4}$ then

$$R(k,\ell) \leq e^{-\frac{\ell}{50}+o(k)} \binom{k+\ell}{\ell}$$

Ideas:

Ideas: "Erdős-Szekeres" Algorithm

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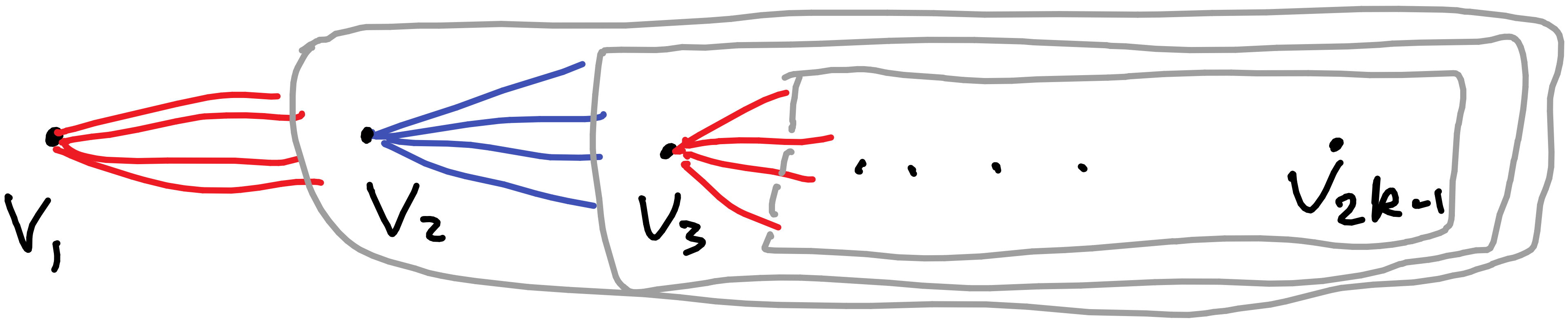
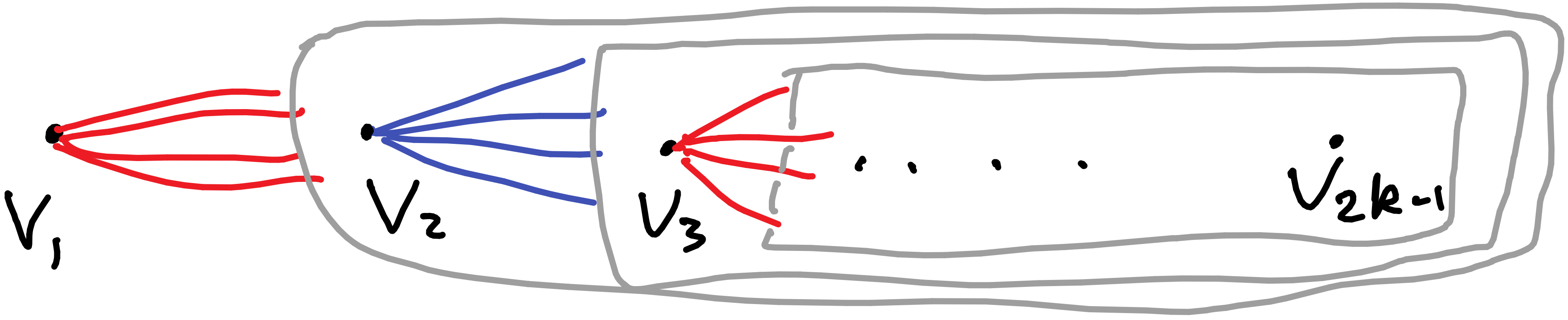
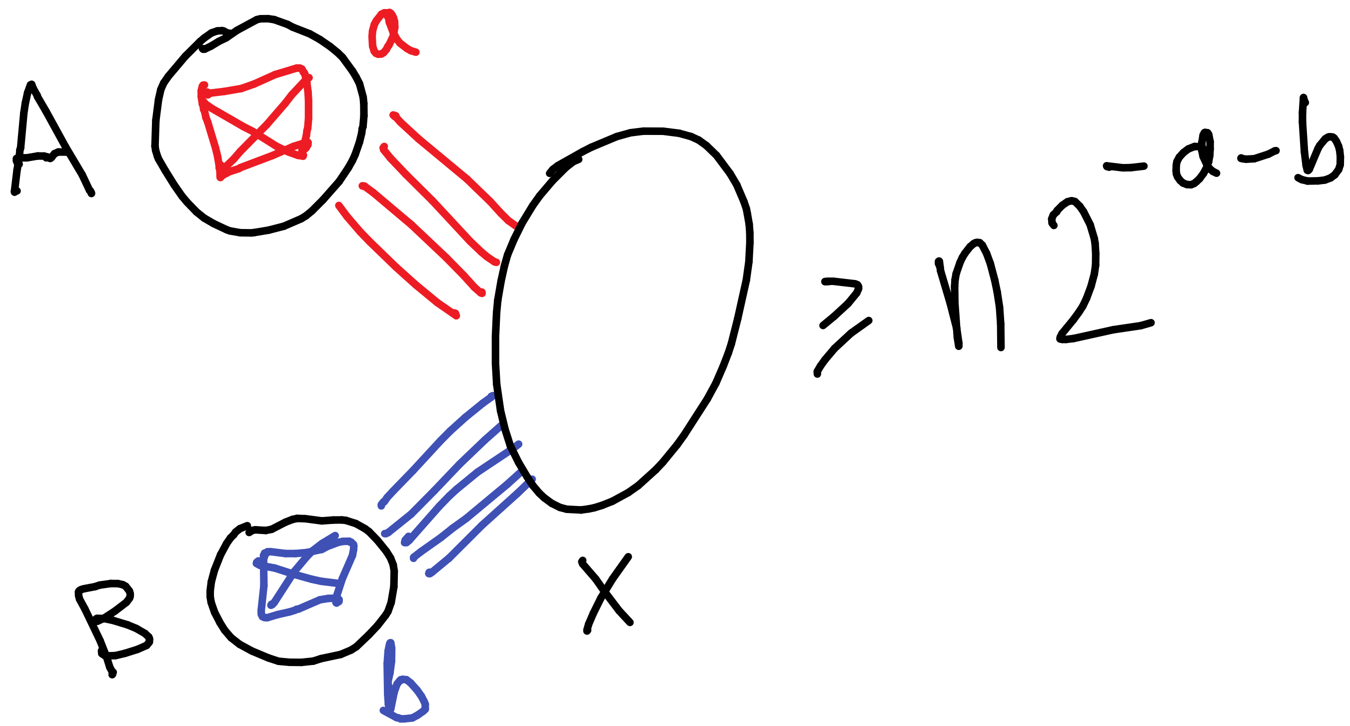
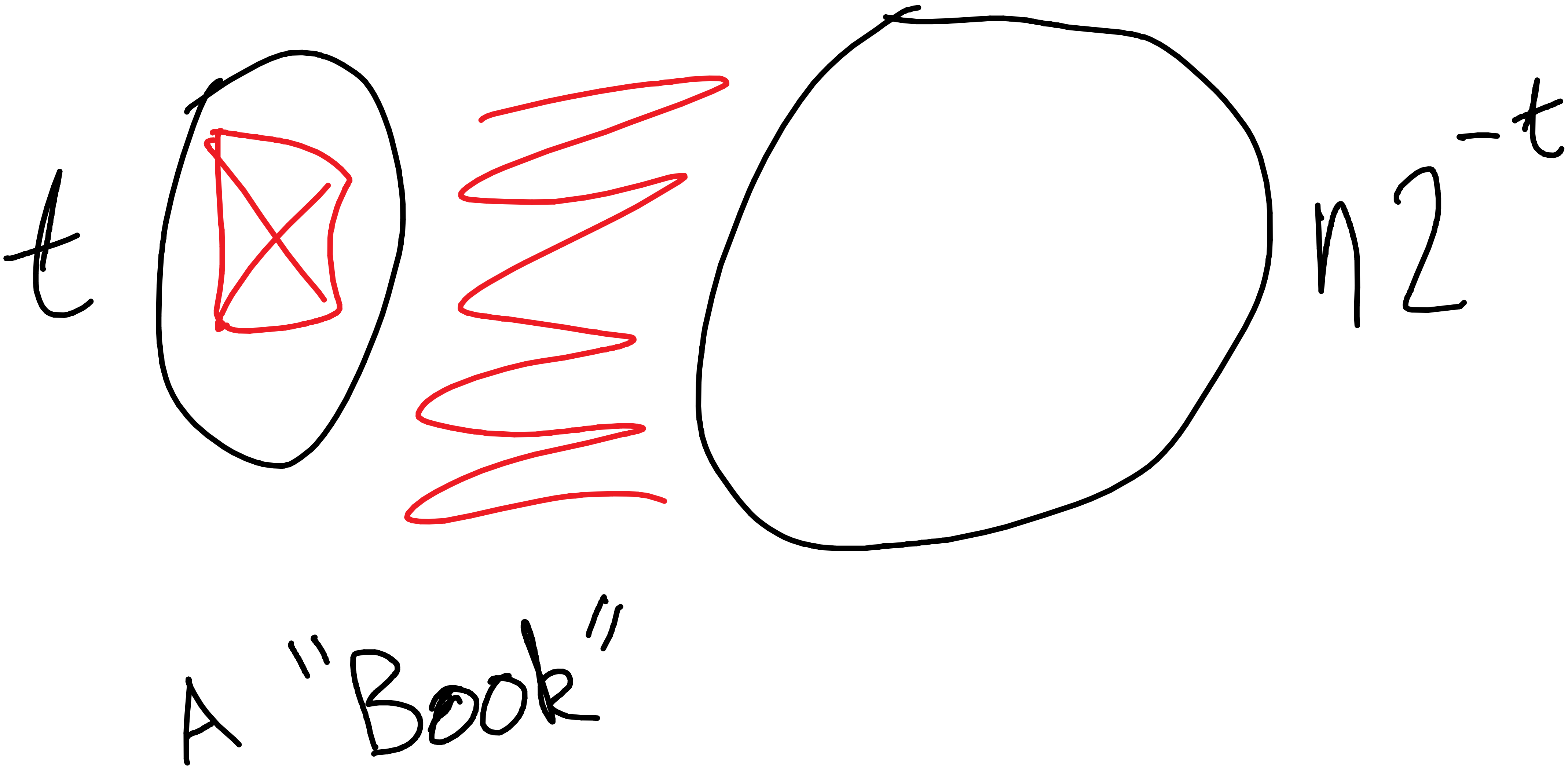


Ideas: "Erdős-Szekeres" Algorithm



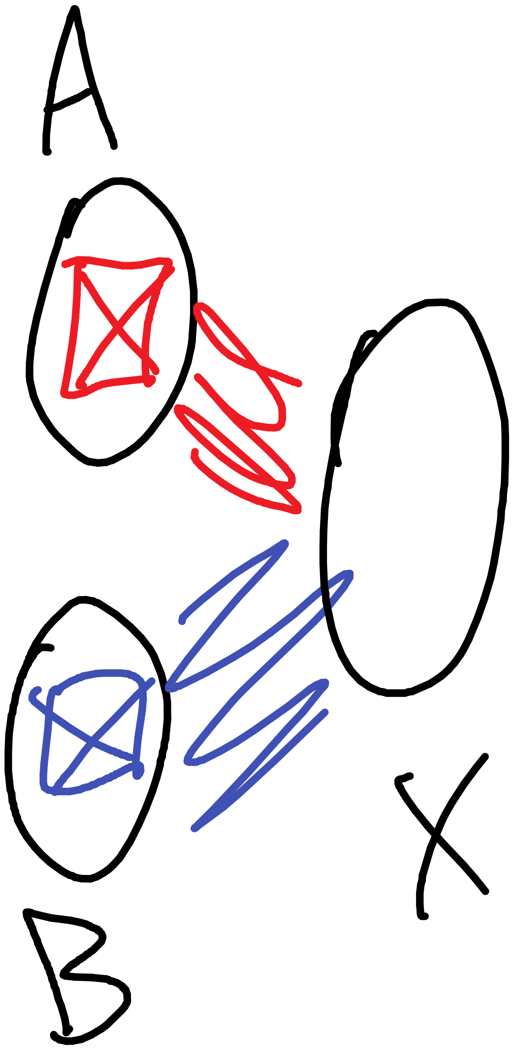


Why not just do red steps?



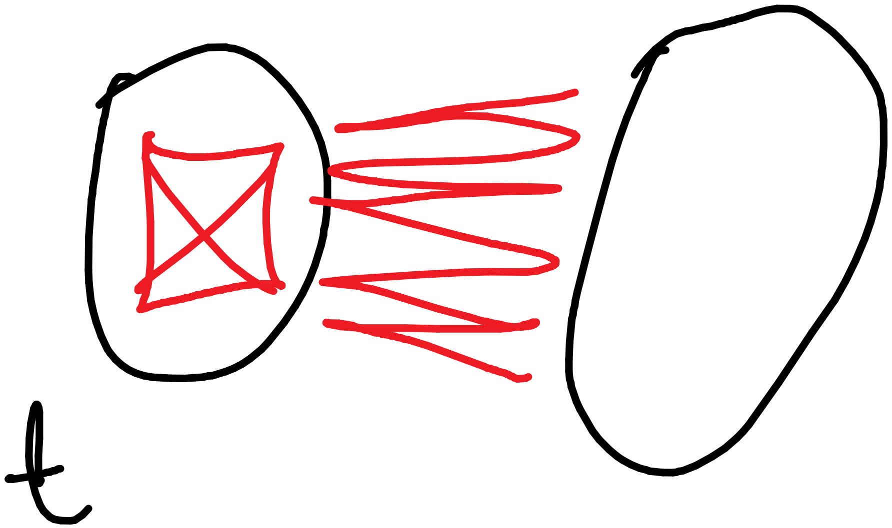
Off-diagonal version:

$$\gamma = \frac{\ell}{k+\ell}$$



$$\geq n(1-\gamma)^{|A|}\gamma^{|B|}$$

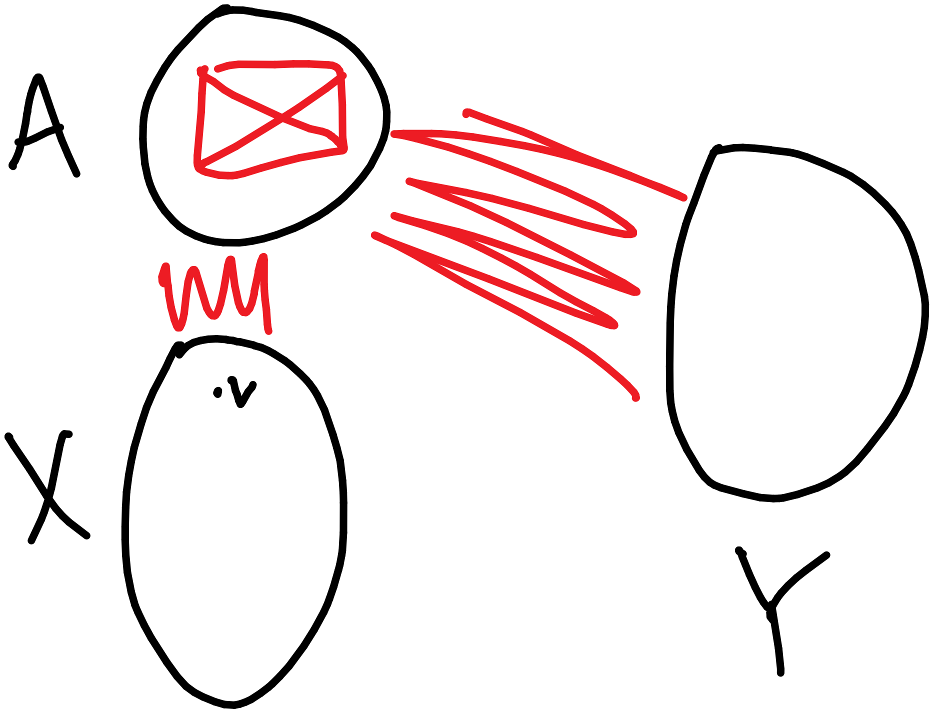
AIM: Find a large red book



$$> \binom{k-t+l}{l}$$

$$> R(k-t, l)$$

NAIVENESS LEVEL 4:



ALWAYS DO RED STEPS

$A \to A \cup \{v\}$

$X \to X \cap N_R(v)$

$Y \to Y \cap N_R(v)$

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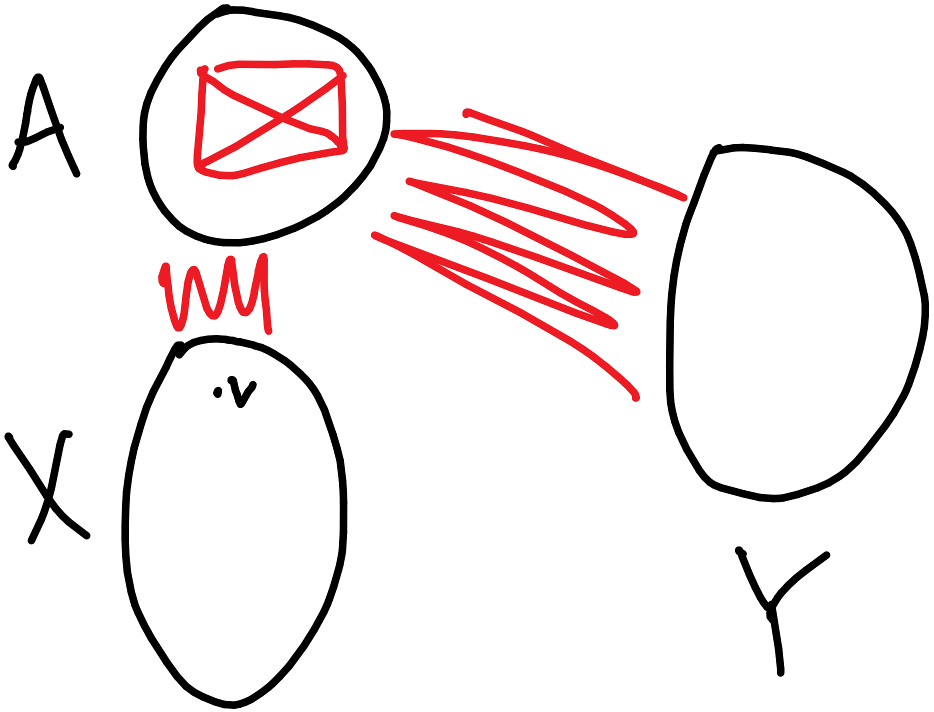

ALWAYS DO RED STEPS

$$A \to A \cup \{v\}$$

$$X \to X \cap N_R(v)$$

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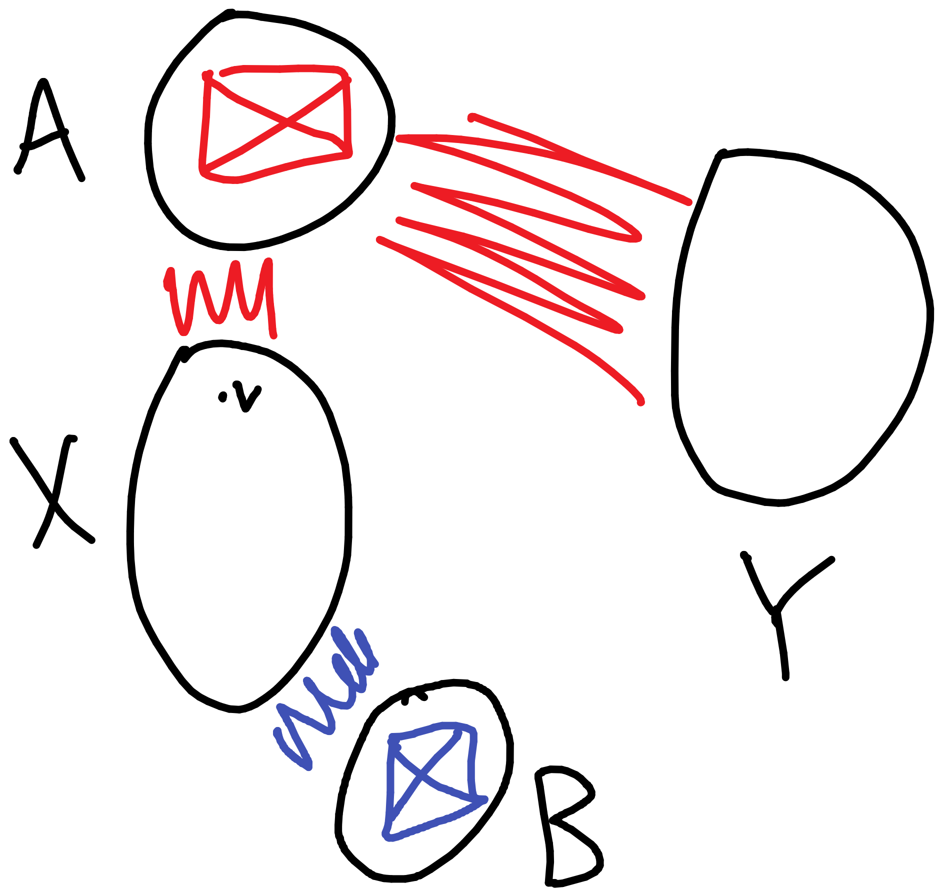
PROBLEM: IF X BECOMES VERY BLUE THEN THE PROCESS ENDS QUICKLY...

NAIVENESS LEVEL 3 : BLUE STEP IF

$|N_B(v) \cap X| \geq \gamma |X|$



$A \to A, \quad B \to B \cup \{v\}$

$X \to X \cap N_B(v)$

$Y \to Y$

ELSE RED STEP

$A \to A \cup \{v\}, \quad B \to B$

$X \to X \cap N_R(v)$

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NAIVENESS LEVEL 3 : BLUE STEP IF

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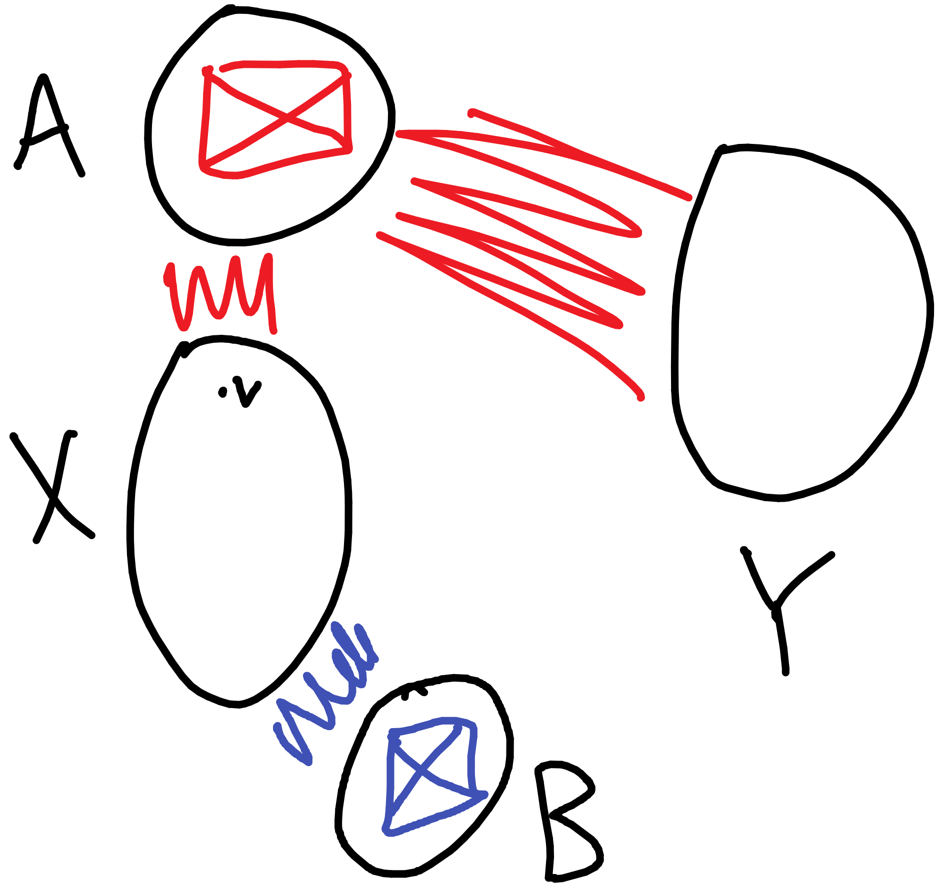

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ELSE RED STEP

$A \to A \cup \{v\}, \quad B \to B$

$X \to X \cap N_R(v)$

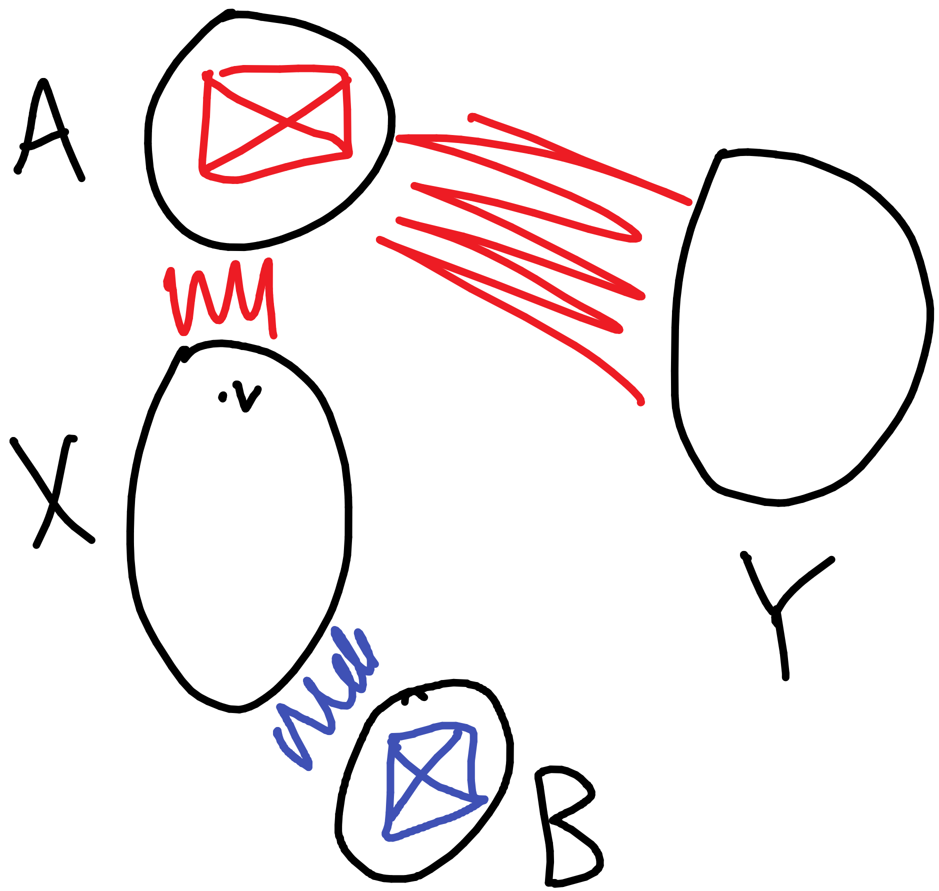
$Y \to Y \cap N_R(v)$

PROBLEM: IN ONE STEP YOU MAY KILL THE BOOK! (IF $N_R(v) \cap Y = \emptyset$)

NAIVENESS LEVEL 2 :

Let $p = \frac{e_R(X,Y)}{|X||Y|}$

red density



Use previous algorithm but with v selected such that

$$|N_R(v) \cap Y| \geq p|Y|$$

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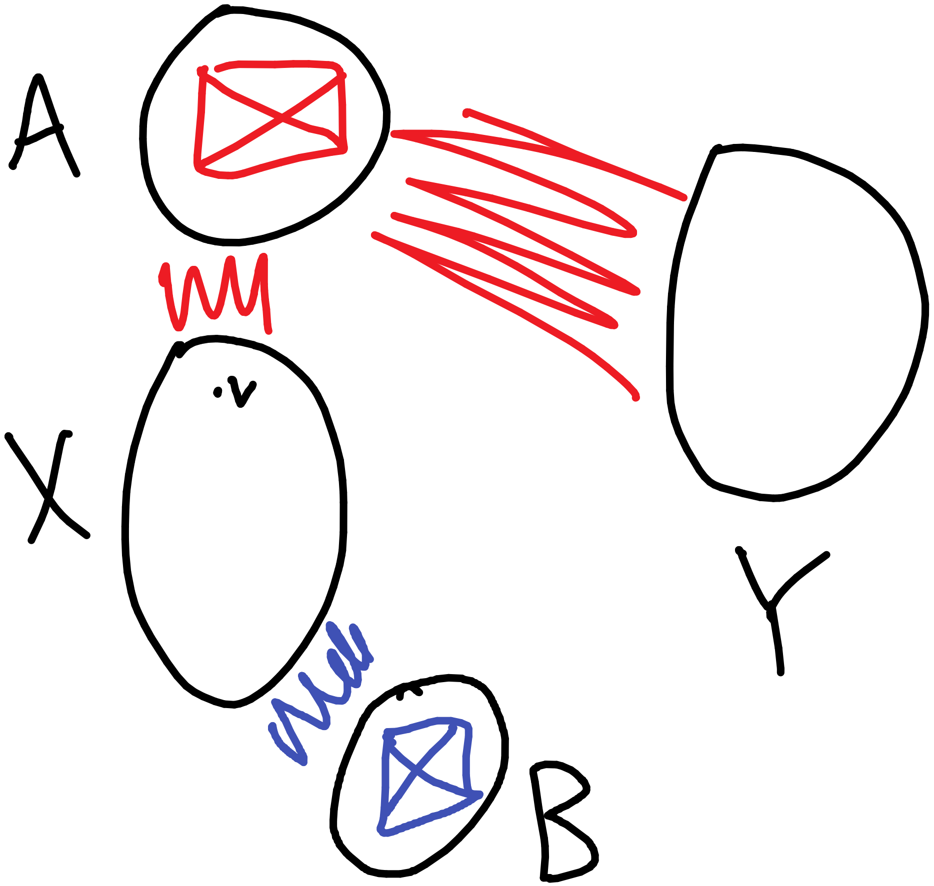
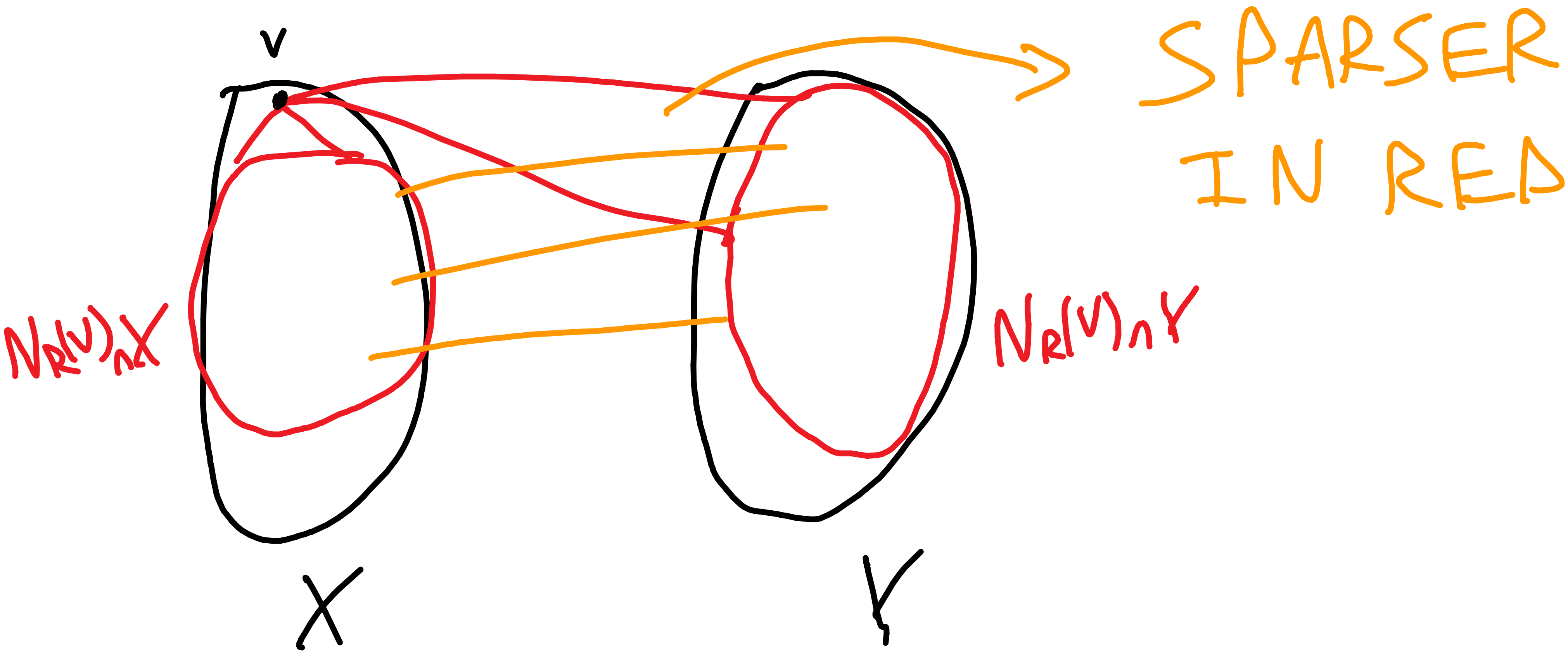


Use previous algorithm but with v selected such that

$$|N_R(v) \cap Y| \geq p|Y|$$

PROBLEM: OVER TIME THE RED DENSITY $p = p$: MAY FALL!

A picture of this problem:



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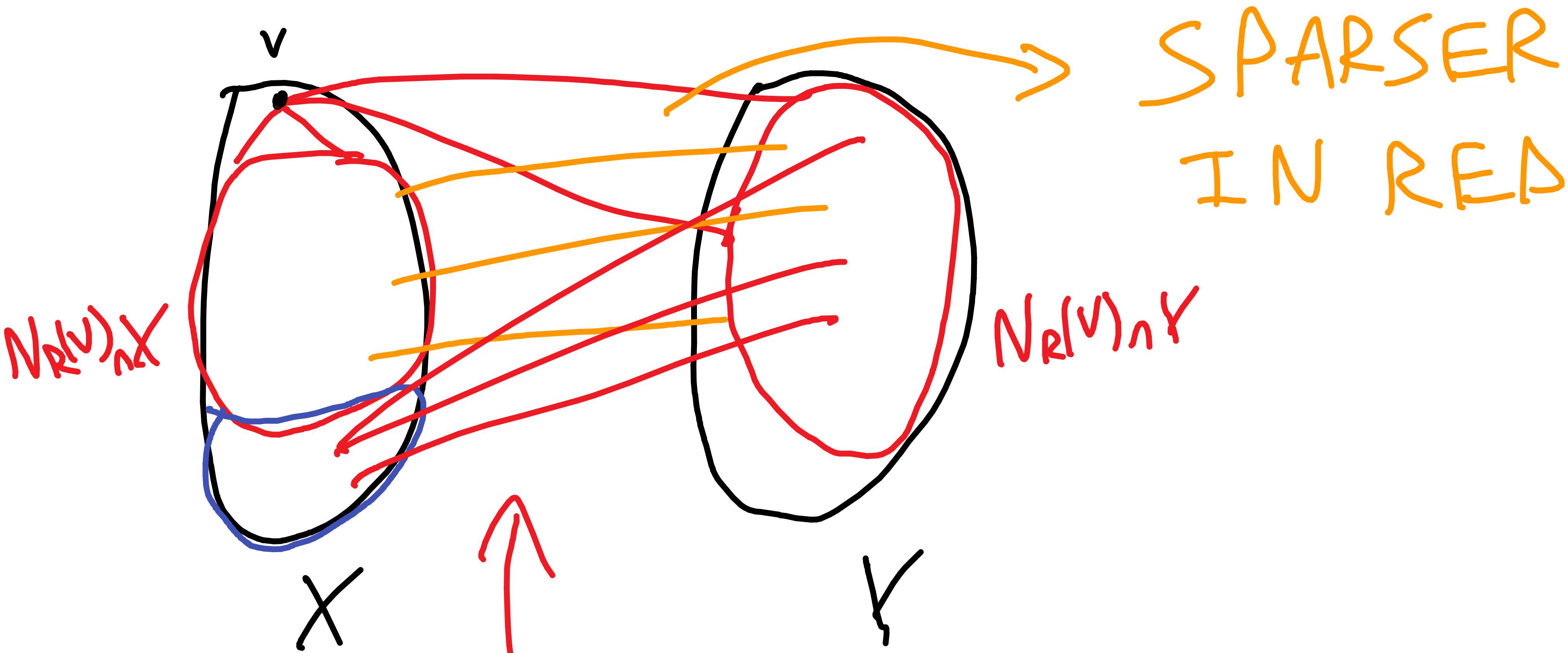

HIGHER RED DENSITY BETWEEN

$N_B(v) \cap X$ AND $N_R(v) \cap Y$!

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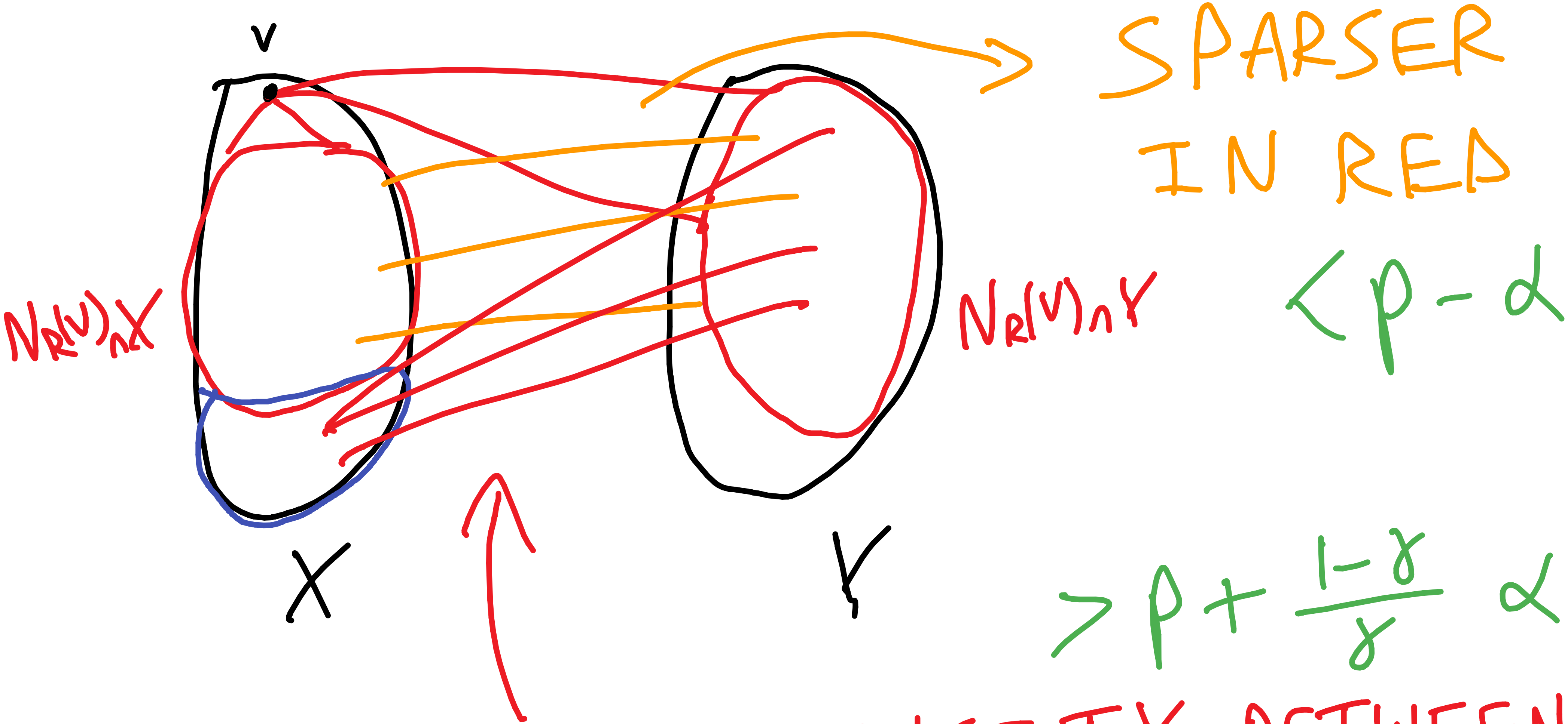

HIGHER RED DENSITY BETWEEN $N_B(v) \cap X$ AND $N_R(v) \cap Y$!

NAIVENESS LEVEL 1 : ALGORITHM

START WITH $A = B = \phi$, X, Y bipartition with

$$p_0 = \frac{e(X,Y)}{|X||Y|} \geq 1 - \delta$$

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↳ DEGREE REGULARISATION STEP*

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↳ IF $\exists v \in X$ $|N_B(v) \cap X| \geq \gamma |X|$ THEN DO A BLUE STEP*

NAIVENESS LEVEL 1 : ALGORITHM

START WITH $A = B = \emptyset$, X, Y bipartition with $p_0 = \frac{e(X,Y)}{|X||Y|} \geq 1-\gamma$

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↳ IF $\exists v \in X$ $|N_B(v) \cap X| \geq \gamma |X|$ THEN DO A BLUE STEP*

↳ ELSE SELECT $v \in X$ SUCH THAT RED DENSITY BETWEEN X AND $N_R(v) \cap Y$ IS $\geq p_i$

↳ IF RED DENSITY BETWEEN $N(v) \cap X$ AND $N(v) \cap Y$ IS $\geq p_i - \alpha$ THEN DO A RED STEP

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↳ IF RED DENSITY BETWEEN $N(v) \cap X$ AND $N(v) \cap Y$ IS $\geq p_i - \alpha$ THEN DO A RED STEP

↳ ELSE DO A DENSITY BOOST STEP:

$$A \to A, \quad B \to B \cup \{v\}$$

$$X \to N_B(v) \cap X, \quad Y \to N_R(v) \cap Y$$

PROBLEM: CHOICE OF α ?

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IF ALL STEPS RED WE DON'T WANT TO LOSE TOO MUCH DENSITY:

TAKE $\alpha k \ll 1$

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IF ALL STEPS RED WE DON'T WANT TO LOSE TOO MUCH DENSITY:

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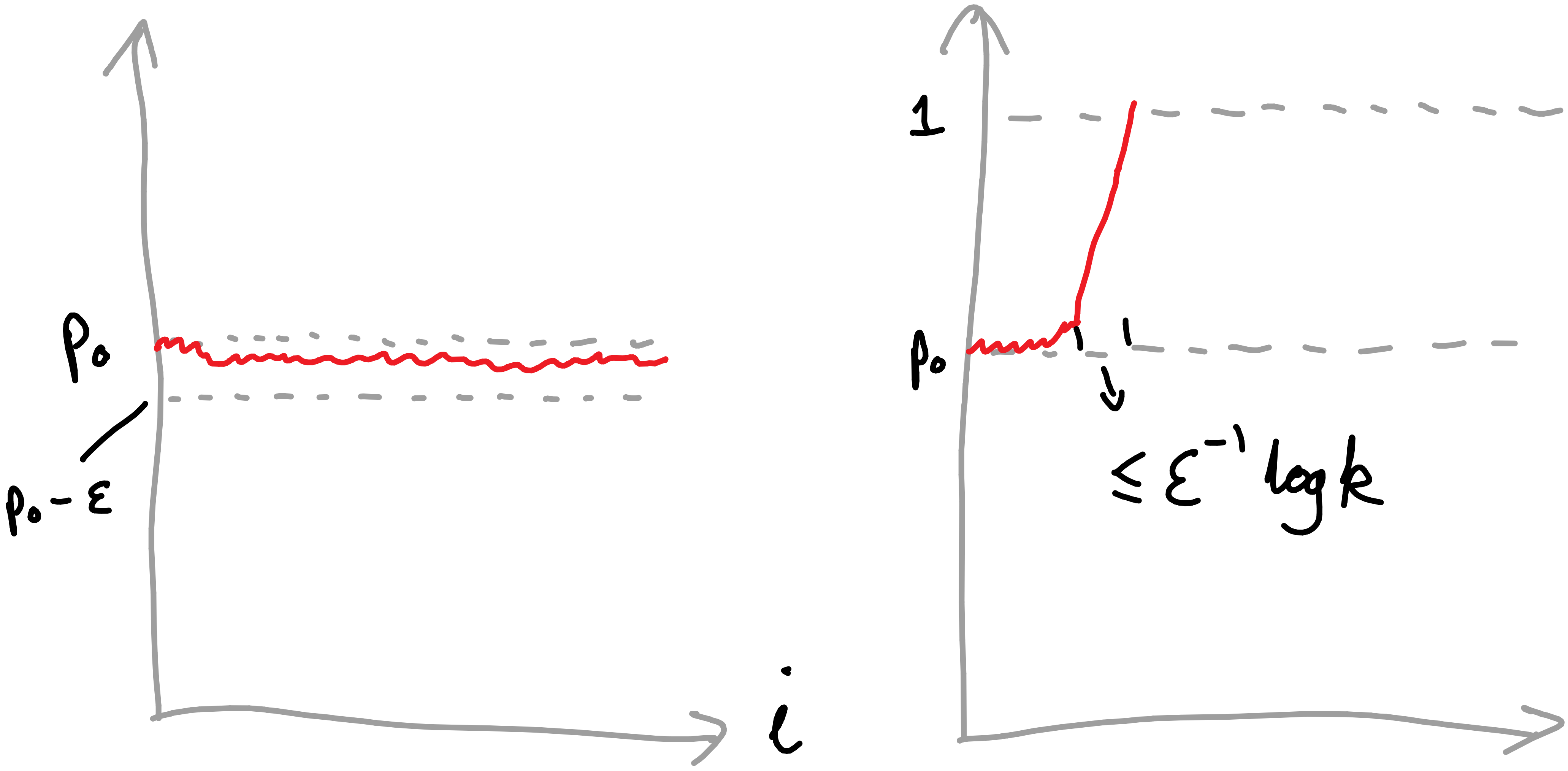
WE CANNOT ALLOW TOO MANY DENSITY BOOST STEPS

TAKE $\alpha k \gg 1$

SOLUTION: $\alpha_i = \frac{\varepsilon}{k} + \varepsilon(\alpha - p_0)$



Some Calculations: After t red steps
b blue steps
and s boost steps

We have

$$|X| \geq 2^{o(k)} (1-\gamma)^t \gamma^{b+s} n$$

$$|Y| \geq 2^{o(k)} (1-\gamma)^{s+t} n$$

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$$|Y| \geq 2^{o(k)} (1-\gamma)^{s+t} n$$

Also, $s \leq \frac{\gamma}{1-\gamma} t + o(k)$

If $\gamma > 0$ small then one may deduce

$\Rightarrow$ $t \approx k$ at the end $\left(\text{even if } n = e^{-\delta \ell} \binom{k+\ell}{\ell} \right)$

$\Rightarrow$ $|Y| \geq \binom{k-t+\ell}{\ell}$

Finally... a conjecture...

Let X_d be a sequence of random vectors with $X_d \in \mathbb{R}^d$.

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Let $X^{(d)}$ be a sequence of random vectors with $X^{(d)} \in \mathbb{R}^d$.

<u>Definitions</u> (1) X is <u>isotropic</u> if $\mathbb{E}[X_i X_j] = \delta_{ij}$

(2) X is <u>sub-gaussian</u> if $\exists c > 0 \ \forall \theta, t$

$$\mathbb{P}\left(\langle X, \theta \rangle \geq t d^{\frac{1}{2}}\right) \leq e^{-ct^2}$$

(3) X is <u>unbalanced</u> if $\mathbb{P}\left(\langle X, X' \rangle \leq -c d^{\frac{1}{2}}\right) \geq \frac{2}{3}$

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__Conjecture__ There does not exist a sequence $X^{(d)} \in \mathbb{R}^d$ which is isotropic, sub-gaussian & unbalanced.

THANK YOU!

OBRIGADO!