Limit Theorems for the simplest parking process

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Ongoing research:
Cooperation between FAPESP and Universidad de Antioquia (Medellin)

In colaboration with

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Many thanks to Eulalia and Giulio for the oportunity to share!

Content

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$\Rightarrow X_{\Lambda_n}$ is called the jamming limit of Λ_n.

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2. Let
$$N_n := \sum_{i \in \Lambda_n} X_{\Lambda_n}(i) \quad \text{and} \quad N_n^Y := \sum_{i \in \Lambda_n} Y(i)$$
What about the statistical properties of X_{Λ_n} and Y?:
$\rightarrow$LLN, TCL, LIL... for N_n and N_n^Y

Interesting model because:

Interesting model because:

- Peculiar type of dependence between the $X_{\Lambda_n}(i)$'s
- It is not defined through conditioning (specifications of statistical physics)
- Strongly non-Gibbsian (for those who know what it takes to be Gibbsian).
- Irreversibility of the dynamics.

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Note that

$$N_{n-1} = \frac{1}{2} Z_n$$

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Alternative perspective:

Particles arrive at random locations, and each adsorbed particle occupies a region of the substrate which prevents the adsorption of any subsequently arriving particle in an overlapping surface region.

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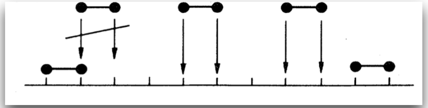


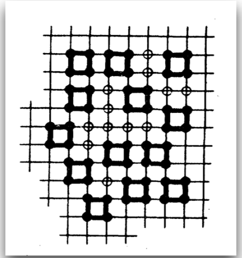
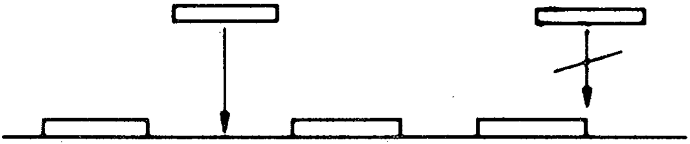


Figure: 2-mers on the left, 2x2-mers on the right

Parenthesis: Continuous counterparts

The Rényi car parking problem: Cars are parked uniformly at random in $[0, x], x > 0$

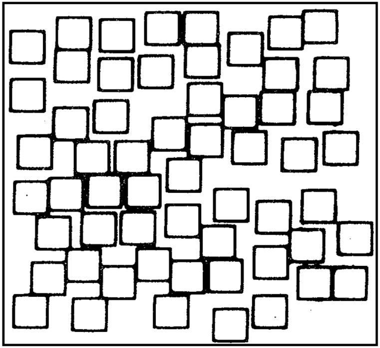


Rényi (1958) proved that

$$\frac{N[0,x]}{x} \to 0.7475979202... \quad a.s.$$

Parenthesis: Continuous counterparts

Cars are parked uniformly at random in $[0, x]^2, x > 0$



(Brosilow *et al.*, 1991) $\lim \frac{N([0,x]^2)}{x^2} \to 0,562009...$ *a.s.*

Other nomenclature/applications/interpretation

- Fatmen seating problem
- Unfriendly seating problem
- Packing problem
- ...

Find applications in

- Polymer chemistry
- Independent sets (graph theory)
- Scheduling problems in operation research
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See the paper by Evans (1993)
"*Random and Cooperative sequential adsorption*"

Some literature (most in 1d)

- Page (1959), Freedman and Shepp (1962), Flajolet (1998), Pinsky (2014), ...

$$\frac{\mathbb{E}(N_n)}{n} = \frac{1}{2}(1 - e^{-2}) + \text{precise error term}$$
$$\frac{\mathrm{Var}(N_n)}{n} = e^{-4} + \text{precise error term}$$

- Page (1959): $\frac{N_n}{n} \xrightarrow{\mathbb{P}} \frac{1}{2}(1 - e^{-2})$
- Penrose (2002) (any dimension): $\frac{N_n}{n} \xrightarrow{L^p} \rho_d$ and CLT.
- Ritchie (2006) (any dimension): Thermodynamic limit and $\frac{N_n}{n} \xrightarrow{a.s.} \rho_d$
- Pinsky (2014) (very fat men): extended results of Page (1959).
- Gerin (2015): didn't know about Ritchie's paper it seems.
- Chern *et al* (2015): "Dinner table".
- And many others papers in Physics literature based on simulations.

Much more related to our problem

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He obtains CLTs for general models, but through a very long and complicated path:

> *"However, since we always obtain our systems* **by taking the random input to come only from inside the target region, rather than restricting a stationary random field to the target region**, *general CLTs such as that of Bolthausen (1982) are not directly applicable."*

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- To use classical results from random field literature: needs a **stationary random fields** on $\mathbb{Z}^d$

$$(Y(i))_{i\in\mathbb{Z}^d}\ ,\quad Y(i) \in \{0,1\}$$

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- That if $\Lambda \subset \Lambda'$, then

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- To use classical results from random field literature: needs a **stationary random fields** on $\mathbb{Z}^d$

$$(Y(i))_{i\in\mathbb{Z}^d}\ ,\quad Y(i) \in \{0,1\}$$

- Satisfying the rules of RSA!

Ritchie constructed such a random field

Thomas Ritchie:

- *Construction of the Thermodynamic Jamming Limit for the Parking Process and Other Exclusion Schemes on* $\mathbb{Z}^d$. (2006).

He proved:

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- *Construction of the Thermodynamic Jamming Limit for the Parking Process and Other Exclusion Schemes on $\mathbb{Z}^d$.* (2006).

He proved:

- Perfect simulation algorithm of Y on any $\Lambda \subset \mathbb{Z}^d$:

$$Y(i) = [f(U)](i) \ , \ \forall i \in \mathbb{Z}^d$$

where

$$U = (U(i))_{i \in \mathbb{Z}^d} \text{ is i.i.d. } U_i \sim \text{Unif}[0,1]$$
$$f : U \to \{0,1\}^{\mathbb{Z}^d} \text{ is translation equivariant.}$$

As a consequence of the construction, he gets:

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- Strong law of large numbers

$$\frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} Y(i) \stackrel{n \to \infty}{\longrightarrow} \rho_d \ , \ \text{a.s.}$$

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$$\frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} Y(i) \overset{n\to\infty}{\longrightarrow} \rho_d \ , \ \text{a.s.}$$

- With a control of boundary effects he proved

$$\frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} X_{\Lambda_n}(i) \overset{n\to\infty}{\longrightarrow} \rho_d \ , \ \text{a.s.}$$

Content

About the random field Y: asymptotic results

Theorem

For any $d \geq 1$, the random field Y satisfies

$$\text{(CLT)} \quad \frac{N_n^Y - |\Lambda_n|\rho_d}{\sqrt{\sigma^2|\Lambda_n|}} \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1)$$

$$\text{(LIL)} \quad \limsup_n \frac{N^Y - |\Lambda_n|\rho_d}{\sqrt{2\sigma^2|\Lambda_n|\log\log|\Lambda_n|}} = 1 \quad a.s.$$

where

$$\sigma^2 = \sum_{i\in\mathbb{Z}^d} \operatorname{Cov}\big(Y(\mathbf{0}), Y(i)\big) > 0. \tag{1}$$

About the random field Y: non-asymptotic result

Theorem

For any $\epsilon > 0, n, d \geq 1$

$$\mathbb{P}\left(\left|N_n^Y - \rho|\Lambda_n|\right| > \epsilon\right) \leq e^{\frac{1}{e} - \frac{\epsilon^2}{4eB|\Lambda_n|}} \tag{2}$$

where $B = B(d)$ *is explicit.*

About the sequence $X_{\Lambda_n}, n \geq 1$

Theorem

- *For any* $n, d \geq 1$

$$\left| \mathbb{E} N_n - |\Lambda_n| \rho_d \right| \leq$$
$$\frac{2d(2d-1)^n}{(n+1)!} + (2d)^2 \sum_{k=0}^{n-1} \frac{(2d-1)^k (2(n-k)+1)^{d-1}}{(k+1)!}.$$

- *The LIL holds for the sequence* $X_{\Lambda_n}, n \geq 1$ *in* $d = 1$.

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- *The LIL holds for the sequence $X_{\Lambda_n}, n \geq 1$ in $d = 1$.*

Couldn't get rid of the boundary effects to get the LIL in $d \geq 2$...

Content

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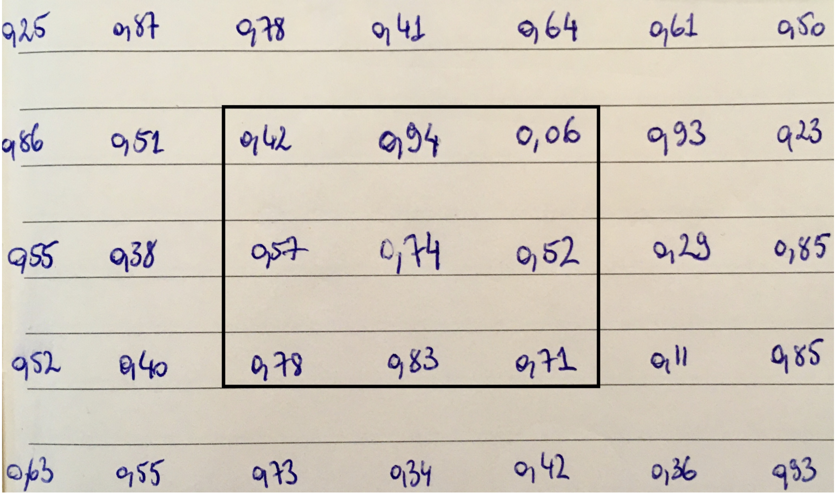
- **Solves the issue of Penrose**
  It simulates from any region $\Lambda \subset \mathbb{Z}^d$ a sample $Y(\Lambda)$ which is a compatible projection of the whole random field $(Y(i))_{i \in \mathbb{Z}^d}$.

- **It gives all we want at once**
  It easily yields good mixing properties allowing to use results from the literature.

- **It is very elegant!**

First step: "the uniforms algorithm" (in $\mathbb{Z}^2$)

Consider $(U_i)_{i \in \mathbb{Z}^2}$ i.i.d.'s with $U_0 \sim \text{Unif}[0, 1]$ and region Λ



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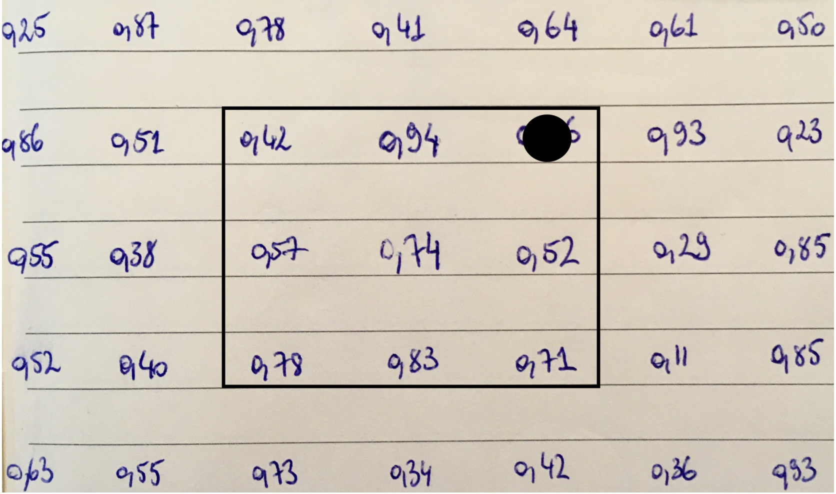

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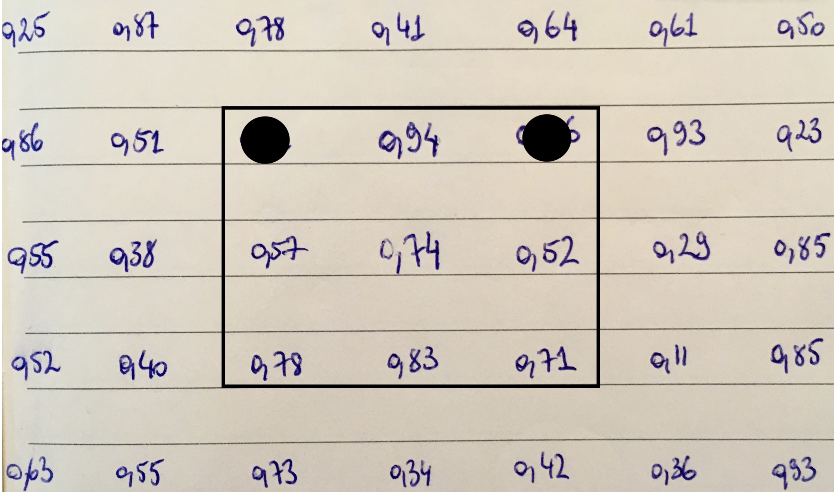

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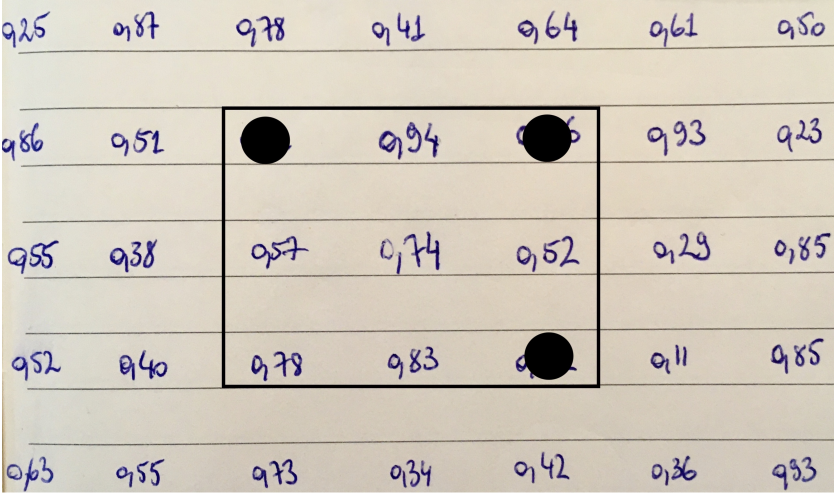

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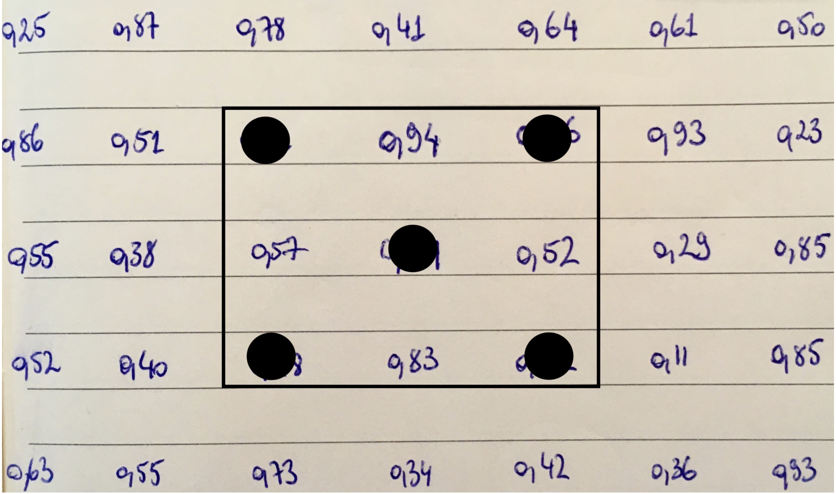

First step: "the uniforms algorithm" (in $\mathbb{Z}^2$)

Observe that:

1. This has exactly the same distribution as the first (definition) algorithm for finite boxes.

2. It appears clearly now why we are not sampling from the thermodynamic limit: look at the 0,11!

3. This last observation is also the key to understanding how the PSA should work:

   The decision of whether or not a particles is put at $i \in \mathbb{Z}^2$ should not depend on the box, **but exclusively on the uniform random variables.**

The perfect simulation algorithm

- Start from $(U_i)_{i \in \mathbb{Z}^2}$ i.i.d.'s with $U_0 \sim \text{Unif}[0, 1]$

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- Define the "armour" of $i \in \mathbb{Z}^2$ by

$$\mathcal{A}(\{i\}) := \bigcup_{y\in\mathbb{Z}^2 : i\to j} \{\text{vertices on the path from } i \text{ to } j\}$$

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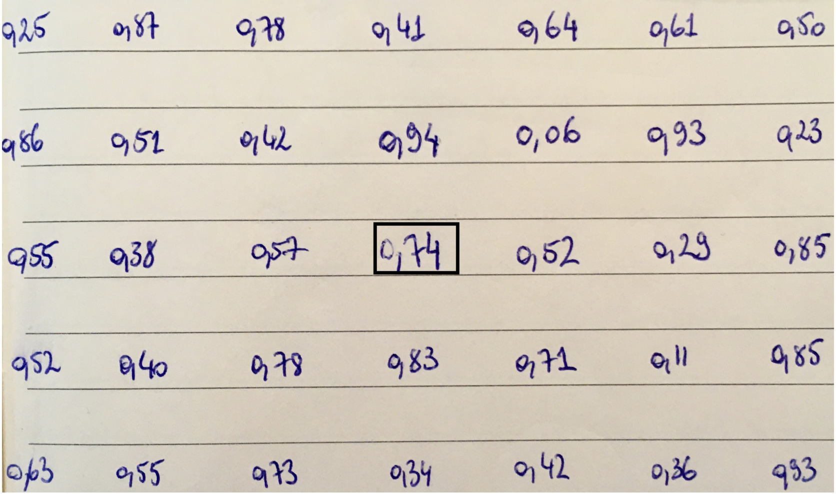
- **The PSA for $i \in \mathbb{Z}^2$:**

Just apply the "uniforms algorithm" in $\mathcal{A}(\{i\})$!!

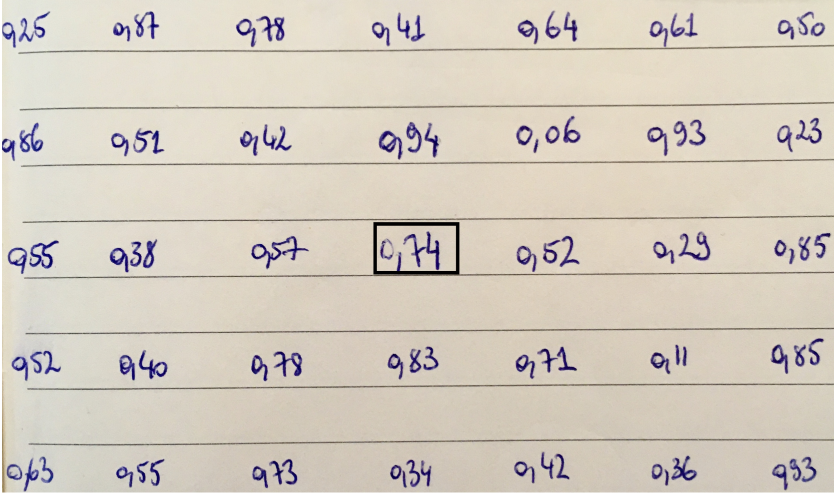
Let us perfectly simulate $Y(\Lambda)$ for $|\Lambda| = 1$

Here is our $\Lambda = \{i\}$...



Let us perfectly simulate X_Λ for $|\Lambda| = 1$

... we construct its armour but going along “decreasing paths”...



0,25	0,87	0,78	0,41	0,64	0,61	0,50
0,86	0,51	0,42	0,94	0,06	0,93	0,23
0,55	0,38	0,57	0,74	0,52	0,29	0,85
0,52	0,40	0,78	0,83	0,71	0,11	0,85
0,63	0,55	0,73	0,34	0,42	0,36	0,33

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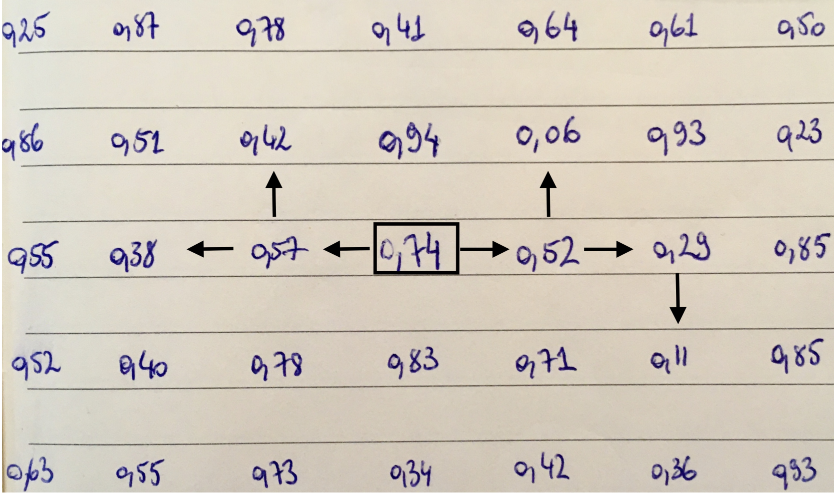

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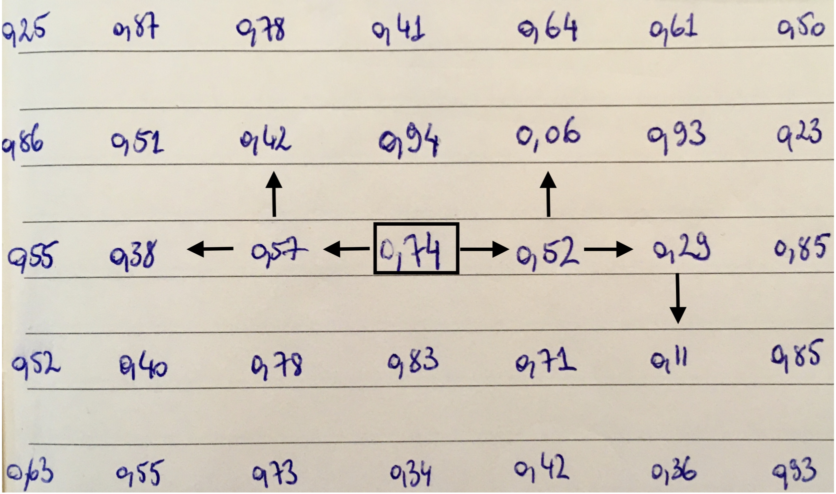

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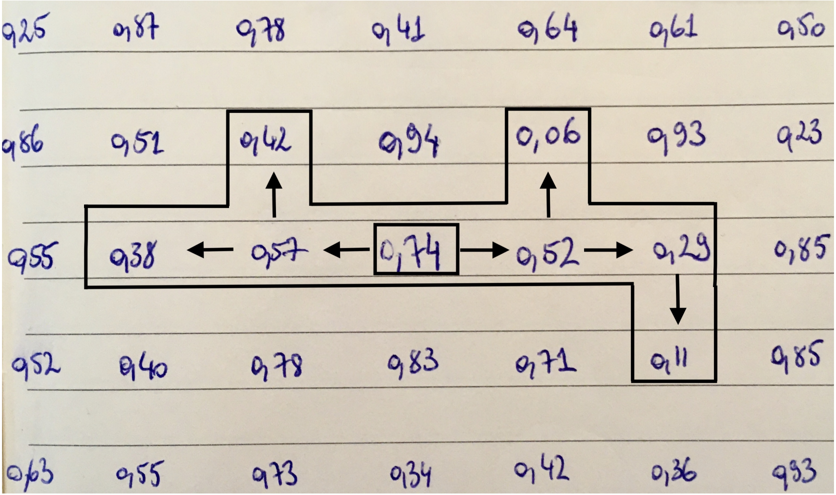

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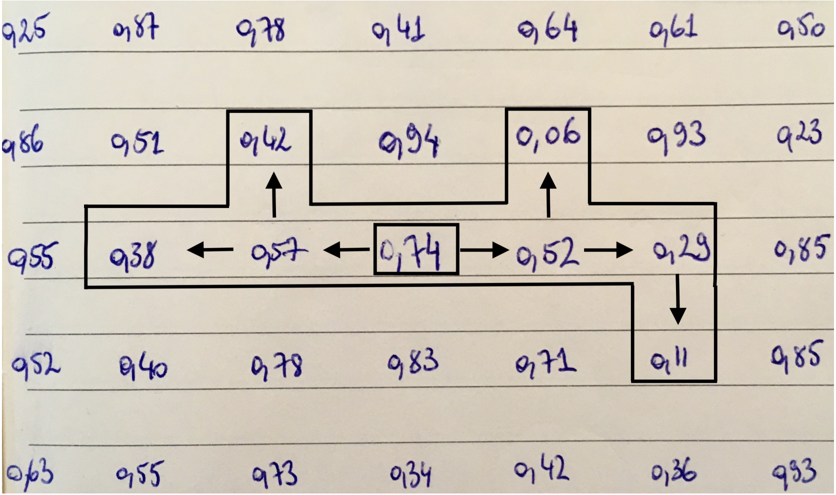
... and we can now use the "uniform algorithm" inside $\mathcal{A}(\{i\})$...



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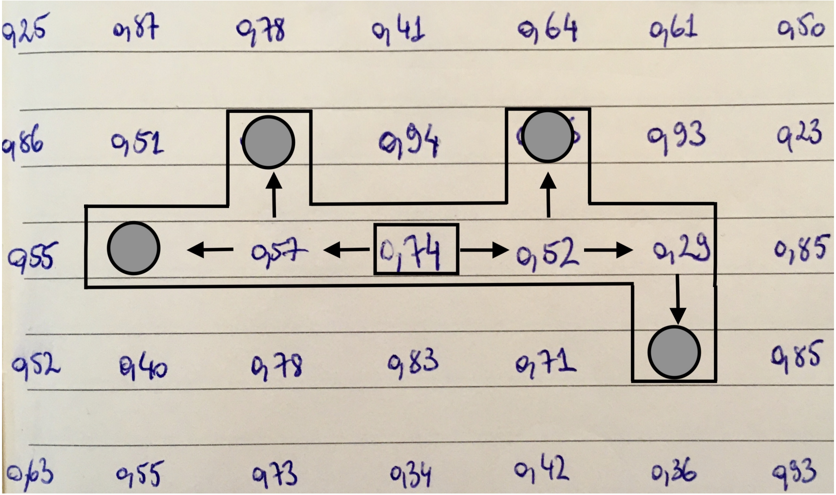
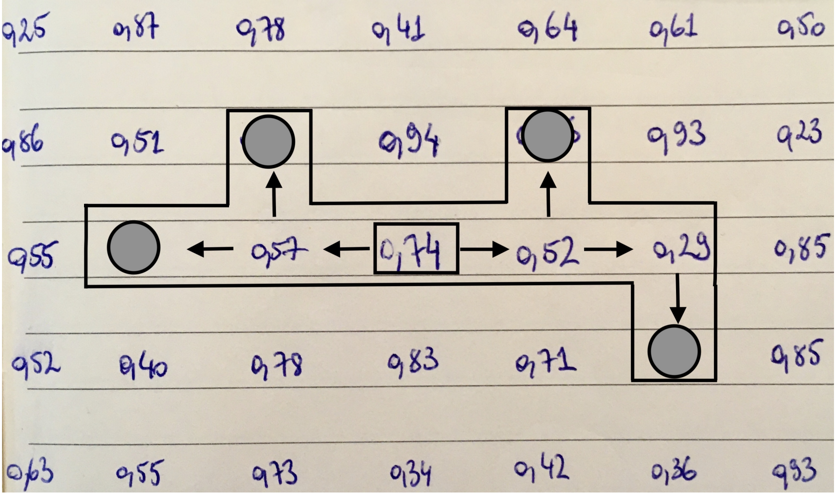


Let us perfectly simulate X_Λ for $|\Lambda| = 1$

... to conclude the algorithm:



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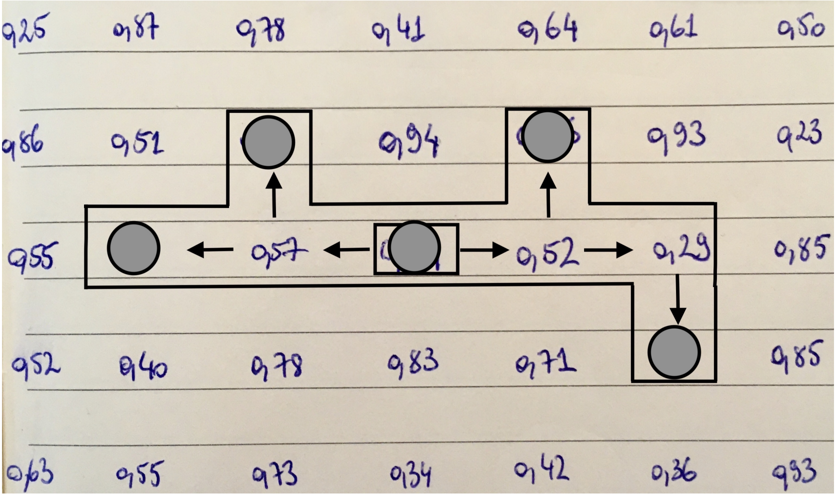
... to conclude the algorithm: $Y(i) = 1$.



For the proofs concerning Y

- We can make the PSA of any finite region $\Lambda \subset \mathbb{Z}^d$ in finite time.
- $\mathcal{A}(\Lambda) = \cup_{i \in \Lambda} \mathcal{A}(\{i\})$
- Moreover $|\mathcal{A}(\{i\})|$ has super-exponential tail.
- This gives very good α-mixing
- $\Rightarrow$ Good mixing implies, for the random field:
  - SLLN,
  - CLT,
  - Berry-Esseen,
  - Concentration inequalities *etc*...

For the proofs concerning $X_{\Lambda_n}, n \geq 1$

- To prove that $\left| \mathbb{E} N_n^Y - |\Lambda_n| \rho \right| \leq F(d)$ observe that

$$\begin{aligned}
&\left| \mathbb{E} N_n^Y - |\Lambda_n| \rho \right| \\
&\qquad \leq \mathbb{E} |N_n^Y - |\Lambda_n| \rho| \\
&\qquad \leq \mathbb{E} \sum_{i \in \Lambda_n} \mathbf{1}\{X_{\Lambda_n}(i) \neq Y(i)\} \\
&\qquad = \sum_{i \in \Lambda_n} \mathbb{P}(X_{\Lambda_n}(i) \neq Y(i)).
\end{aligned}$$

For the proofs concerning $X_{\Lambda_n}, n \geq 1$

- To prove that $\left|\mathbb{E}N_n^Y - |\Lambda_n|\rho\right| \leq F(d)$ observe that

$$\begin{aligned}
&\left|\mathbb{E}N_n^Y - |\Lambda_n|\rho\right| \\
&\qquad \leq \mathbb{E}|N_n^Y - |\Lambda_n|\rho| \\
&\qquad \leq \mathbb{E}\sum_{i\in\Lambda_n} \mathbf{1}\{X_{\Lambda_n}(i) \neq Y(i)\} \\
&\qquad = \sum_{i\in\Lambda_n} \mathbb{P}(X_{\Lambda_n}(i) \neq Y(i)).
\end{aligned}$$

- Proof of the LIL in 1d for $X_{\Lambda_n}, n \geq 1$

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  - We can show that

$$\mathbb{P}(|N_n - \bar{N}_n| > M) \leq 2 \frac{1}{\lceil M/2 + 2 \rceil !}.$$

For the proofs concerning $X_{\Lambda_n}, n \geq 1$

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  - Thus $|N_n - \bar{N}_n| > \sqrt{|\Lambda_n|}$ finitely many times.

Thank you!