# Coin turning and related walks

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# Co-author(s)

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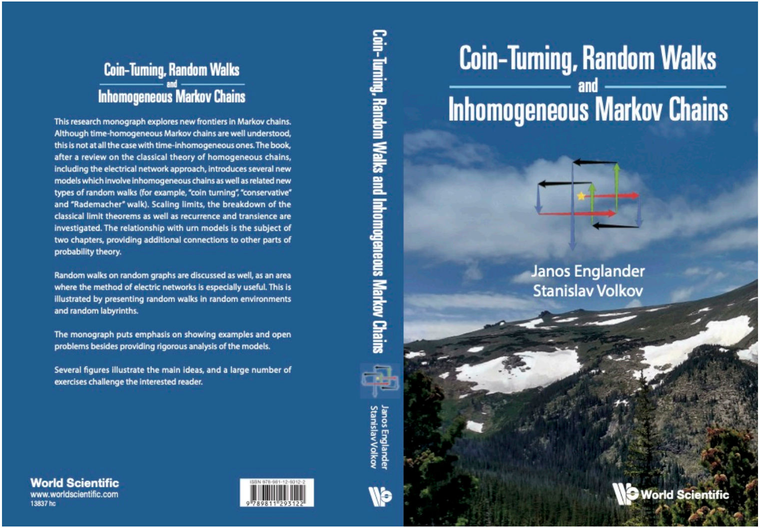
(Some are joint with my former student, Z. Wang.)

# Promotional slide; new book



1. Part one: Turning instead of tossing

2. Part two: the walk

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If we write $+1$ and $0$ for H and T ($X_n = 0, 1$), then this translates into: Relative frequency of heads is 'close' to $1/2$. (Then $Y_n = 2X_n - 1$ and $T_n := X_1 + X_2 + ... + X_n =$ number of H's.)

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(Goes back to Abraham de Moivre, 1738.)

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**Useful representation:** with the indicators of turns $W_k \sim \mathrm{Ber}(p_k)$, one has

$$Y_j = Y_1(-1)^{\sum_2^j W_k}, \ j > 1.$$

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Then by the Borel-Cantelli Lemma,

$$P(\text{Only finitely many turns}) = 1,$$

so

$$P(\text{Limit of rel. frequencies}{=}1) = P(\text{Limit of rel. frequencies}{=}0) = 1/2.$$

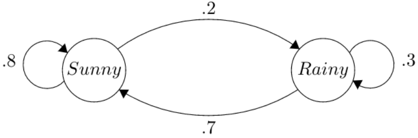
# Intuition

There must be two *phase transitions*, somewhere between the two extremes above:

(a) One, where LLN breaks down.
(b) One where classical CLT (i.e. order $\sqrt{n}$ fluctuations for $S_n$) breaks down.

# Intuition; Markov chains

**Q:** What's the big deal:? It is a 2 state Markov chain!



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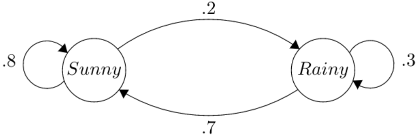


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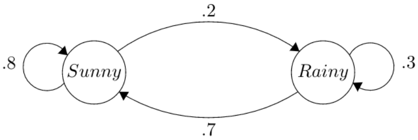


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They are not well studied, except Dobrushin's research in the 1950s and some revival after 2000 (Dietz, Sethuraman, Varadhan, Peligrad).

# Example: Markov chain CLT

**(Ex0)** Let $p_n = c$, where $0 < c < 1$. When $c \neq 1/2$, the outcomes are not independent. Recall that $S_n := Y_1 + Y_2 + ... + Y_n$. One has

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Or, in terms of the frequency of heads,

$$\text{Law}(\sqrt{N}\,(\overline{X}_n - 1/2)) \Rightarrow \text{Normal}\left(0, \frac{1-c}{4c}\right),$$

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The term $\frac{1-c}{c}$ can be arbitrarily large (small) when $c$ is sufficiently small (close to 1) and thus turns occur very rarely (frequently).

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It can still be considered CLT because $n^{(1+\gamma)/2}$ is the order of the standard deviation of $S_n$.

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In terms of $Y_1, ..., Y_n$: the correlation is as strong as in the case of identical variables ($\text{Var}(S_n)$ is of order $n^2$) and the fluctuations of $S_n$ are of order $n$, destroying the LLN.

## Example: Variation of the previous one

**(Ex2b)** Let $p_n = a/n$ with $a > 0$. Again, LLN breaks down, and the proportion is no longer concentrated around $1/2$, and in fact,

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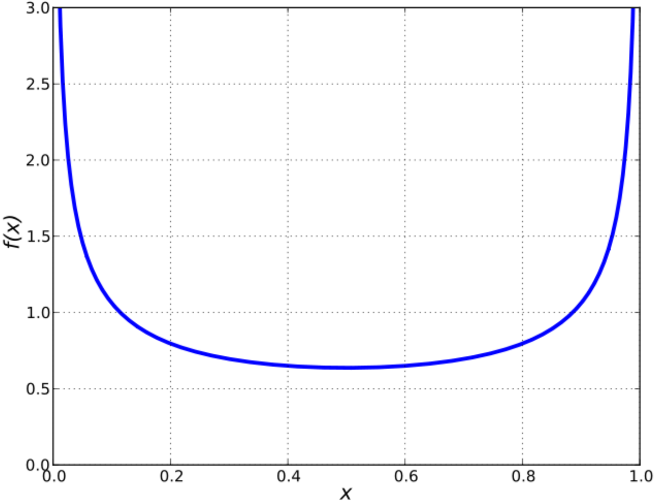
(concave, flat,convex)

# Beta(1/2,1/2)=ArcSine distribution



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It is as far away from $\delta_{1/2}$ (LLN case) as possible!

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Intuitively: small $|e_{i,j}|$ needed to get close to classical limit theorems.
If there is a single $p_k = 1/2$ with $i + 1 \leq k \leq j$ then $Y_i$ and $Y_j$ are uncorrelated. Why?

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Assume that [condition in terms of the $e_{i,j}$ above.]

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**Example:** $p_n = a/n^\gamma$ with $a > 0$ and $0 < \gamma < 1$.

(We also have a condition in the converse direction, guaranteeing the LLN breaks down.)

# Theorem Weak Law of Large Numbers

$S_N/N \to 0$ in probability, i.e. for any $\epsilon > 0$,

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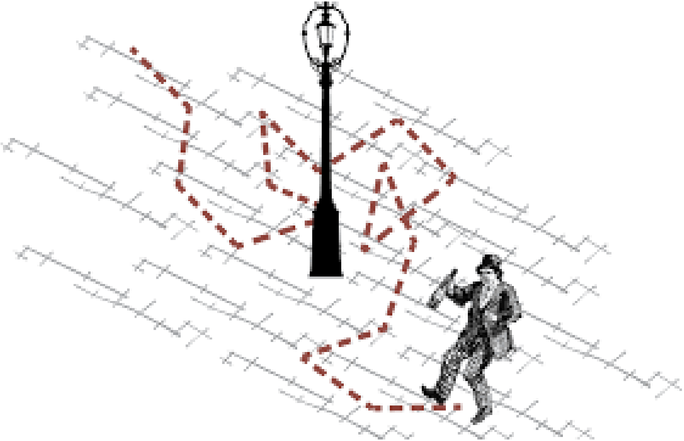
It implies monotonicity for WLLN in terms of the sequence of the $p_n$'s. (How about SLLN??)

1 Part one: Turning instead of tossing

2 Part two: the walk

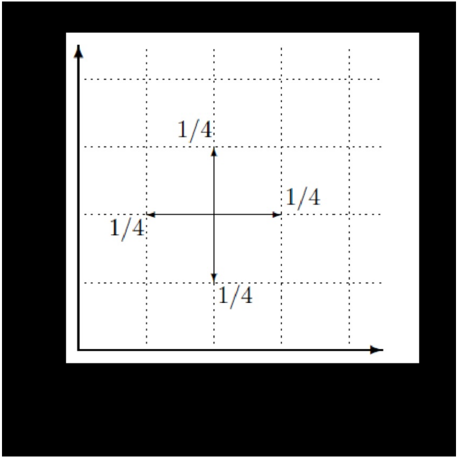
# Random walks (drunkard's walk)

Pólya, 1924.



# Random walks

Classically, steps are independent of the past and each direction has equal probability.

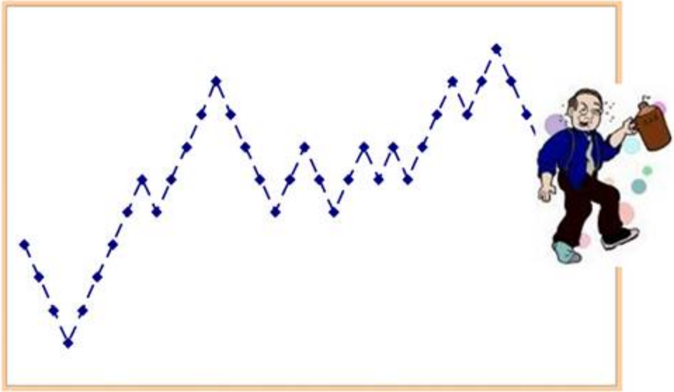


# A different type of RW

## Definition (Coin-turning walk)

CTW: $S_n := Y_1 + ... + Y_n$ for $n \geq 1$; we can additionally define $S_0 := 0$, so the first step is to the right or to the left with equal probabilities.

Extend $S$ to a *continuous time process*, by linear interpolation.



(Here $d = 1$, horizontal axis: time.)

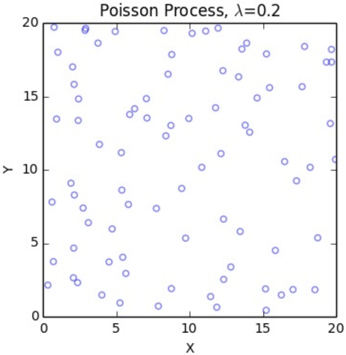
# Cooling cases

Most interesting ("critical") case is when $p_n = a/n$ for $n \geq n_0$ $(a > 0)$.

## Theorem

*Let $S^n$ be defined by $S^n(t) := S_{nt}/n,\ t \geq 0$.*
*(Tiny steps but a lot of them.)*
*In the critical case, $\lim_{n\to\infty} S^{(n)}$ is the zigzag process, where the limit is meant in law.*

# What is a Poisson Point Process?

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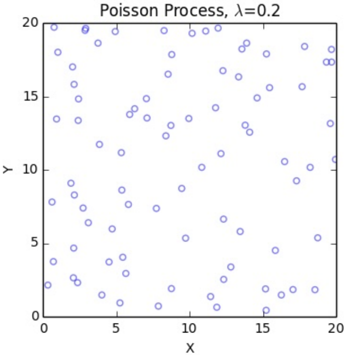


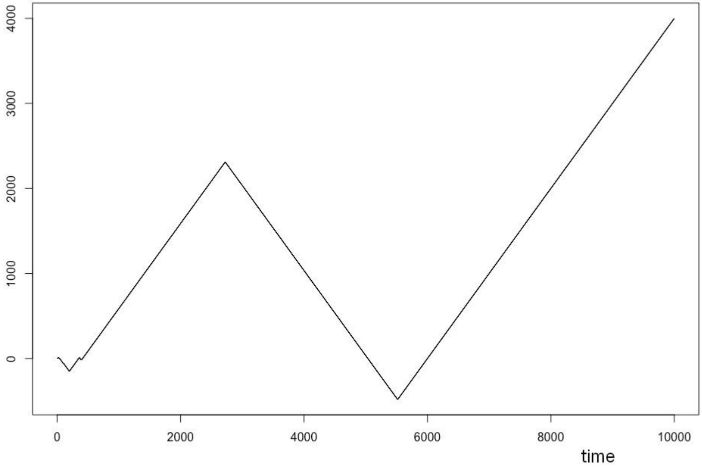

- The number of points in two disjoint sets are independent random variables;
- in one given set their distribution is Poisson with parameter= measure of the set ("intensity measure").

This measure can be given by a density function in $\mathbb{R}$ or in $(0,\infty)$ or in $\mathbb{R}^d$ etc.

For example, on $(0,\infty)$, if the density is $1/x$, then on the interval $(c,d)$ we'll have a Poisson number of points with parameter $\int_c^d \frac{1}{x}\,dx = \log d - \log c = \log(d/c)$, when $c > 0$.

However, on $(0,d)$ the number of points will be infinite, with probability one.

# Simulated pic of the Zigzag process



## 'Zigzag process' description

Consider a Poisson point process (PPP) on $(0, \infty)$ with intensity measure $\frac{a}{x}\,dx$: "scale-free PPP".
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Once the realization is fixed, the value of the process at $t \geq 0$ is obtained as follows.

Starting with the segment containing $t$ and going backwards towards the origin, color the first, third, fifth, etc. segment blue between points. The second, fourth, etc. will be colored red. Given this Poissonian intensity, we'll have infinitely many segments towards zero (and also towards infinity) a.s.

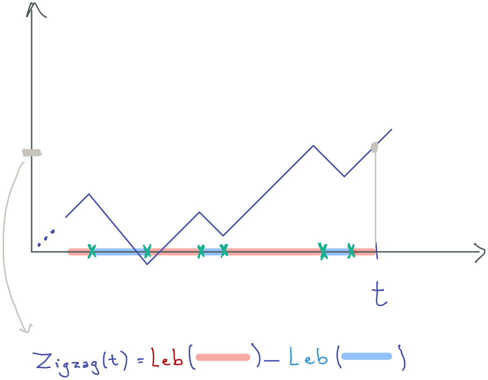
Let $\lambda_b(t)$ and $\lambda_r(t)$ denote the Lebesgue measure of the union of blue, resp. red segments between 0 and $t$. Then

$$\text{Zigzag}_t := W[\lambda_b(t) - \lambda_r(t)],$$

where $W$ is a random sign, that is $W = -1$ or $W = 1$ with equal probabilities.

# A picture is worth a thousand words

How does one get the zigzag path from the realization of the PPP(c/x), assuming increase at t?



## Localization and recurrence for the walk S

### Definition (Range and effective range)

The *range* of the process $S$ is

$$\mathcal{R}_0 := \{x \in \mathbb{Z} : \ S_n = x \text{ for some } n\}.$$

The *effective range* of the process $S$ is

$$\mathcal{R} := \{x \in \mathbb{Z} : \ S_n = x \text{ for infinitely many } n\} \subseteq \mathcal{R}_0.$$

Note that either $\mathcal{R} = \emptyset$ or $\mathcal{R}$ is a connected set in $\mathbb{Z}$.

### Definition (Localization, recurrence)

The walk *localizes* or *gets stuck* if $\mathcal{R}$ is non-empty and finite ($\Rightarrow \mathcal{R}_0$ is finite). We call $S$ *recurrent*, if $\mathcal{R} = \mathbb{Z}$. ($\mathcal{R} = \emptyset$ means $S_n \to \infty$.)

# Dichotomy

## Theorem (Dichotomy under mixing)

*The mixing condition implies that either*

$$\mathbb{P}(S \textit{ is recurrent}) = 1 \quad \textit{or} \quad \mathbb{P}(S \textit{ gets stuck}) = 1.$$

What is mixing??

# Mixing

We now *do not* randomize the walk with taking $Y_1$ to be symmetric. Yet, it is still true for the indicators of turns $W_k$, that $Y_j = Y_i(-1)^{\sum_{i+1}^{j} W_k}$, $j > i$, and that for $e_{i,j} = \prod_{k=i+1}^{j}(1-2p_k)$ we have $\mathbb{E}(Y_j \mid Y_i) = Y_i\,\mathbb{E}(-1)^{\sum_{i+1}^{j} W_k} = e_{i,j}Y_i$, hence $\mathbb{E}(Y_iY_j) = e_{i,j}$. The sequence $(Y_n)_{n\geq 1}$ satisfies the mixing condition if

$$\lim_{j\to\infty} e_{ij} = 0, \forall i \in \mathbb{N}. \tag{1}$$

Under mixing, one has that $\lim_{j\to\infty}\mathbb{E}(Y_j \mid Y_i) = 0$, so $Y_j$ "becomes symmetrized" for $i$ fixed and large $j$. Also, $\lim_{j\to\infty}\mathbb{E}(Y_i\,Y_j) = 0$ and $\lim_{j\to\infty}\mathbb{E}\,Y_j = 0$, hence, for fixed $i \geq 2$,

$$\lim_{j\to\infty} \mathrm{Cov}(Y_j, Y_i) = 0, \tag{2}$$

in accordance with the usual notion of mixing.

# Criterion for mixing

Mixing has a simple characterization in terms of the sequence $\{p_n\}$:

**Lemma (Criterion for mixing)**

*Mixing holds if and only if*

$$\sum_n \min(p_n, q_n) = \infty. \tag{3}$$

## Condition for recurrence

We discuss just one representative theorem:

**Definition (Spreading)**

$S$ has the spreading property if $|S_n| \to \infty$ as $n \to \infty$ in probability.

**Theorem (Spreading and mixing is enough)**

*Assume spreading and also mixing. Then $S$ is recurrent.*

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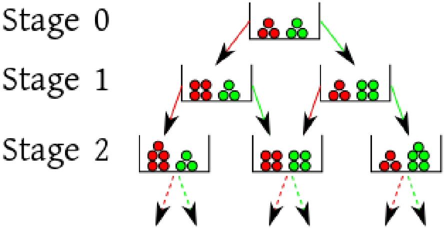
This yields recurrence in a very broad range of models (sequences).

# Pólya urn

Another cool connection in the critical case is with **Pólya urns**.




Observation leading to the connection: if the sequence is precisely $(p_1 = 1/2), p_2 = 1/3, p_3 = 1/4, p_4 = 1/5, \dots$ , then $\frac{S_N}{N}$ has precisely *discrete uniform law* for each $N$. (So no wonder the limit is uniform on $[0, 1]$.)


*Thank you for your attention!*

# Mimicking — if time permits

Let $X = \{X_i\}_{i\geq 1}$ be a sequence of integer-valued RVs under the law $P$. Consider $T$ defined by $T_n := \sum_1^n X_i \ n \geq 1$ the corresponding walk. Let $X^{'} = \{X_i^{'}\}_{i\geq 1}$ be another sequence of integer-valued RVs under $\mathbf{P}$ and let $T_n' := \sum_1^n X_i'$ be the corresponding walk.

## Definition (Mimicking)

We say that $X'$ *mimics* $X$ (and vice versa) if

(M1) $P(X_i = u) = \mathbf{P}(X_i^{'} = u), \ i \geq 1;$

(M2) $P(T_{i+1} = v \mid T_i = u) = \mathbf{P}(T_{i+1}^{'} = v \mid T_i^{'} = u), \ i \geq 1.$

That is, $X$ and $X^{'}$ have the same one dimensional marginals and the two walks have the same one-step transition probabilities.

# Coin vs. Pólya

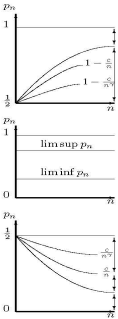
| | Steps | Walk |
|---|---|---|
| **Coin turning** | Markovian, non-exchangeable | non-Markovian |
| **Pólya urn** | non-Markovian, exchangeable | Markovian |

Table 1: Different properties.

## Theorem

*The coin turning process $X$ mimics the urn process $X^{urn}$.*

# Various regimes




HEATING

$\sum q_n < \infty$, beyond mixing

Brownian motion: $q_n \to 0$, and $I = II = \infty$, and $q_{2m} > C \max_{l \geq m} q_{2l+1}$ or $q_{2m+1} > C \max_{l \geq m} q_{2l}$

NEITHER HEATING NOR COOLING

$0 < \liminf_{n\to\infty} p_n \leq \limsup_{n\to\infty} p_n < 1$, Brownian motion

COOLING

subcritical: $np_n \uparrow \infty$, Brownian motion

critical: $np_n \to c > 0$, zigzag

strongly critical : $np_n \to 0$, $\sum p_n = \infty$, ray

supercritical: $\sum p_n < \infty$, beyond mixing