Structural results for the Tree Builder Random Walk

Giulio Iacobelli
Federal University of Rio de Janeiro (UFRJ), Brazil

Joint work with

János Engländer
University of Colorado Boulder - US

Rodrigo Ribeiro
University of Denver - US

Gábor Pete
Alfréd Rényi Institute of Mathematics - Hungary

The Barabási-Albert model

- The preferential attachment random graph model has been proposed by Barabási and Albert in 1999
- Explains the formation of scale-free networks, characterized by a few highly connected nodes
- The model is based on the principle that new nodes are more likely to connect to already well-connected nodes, leading to “hubs”
- Its rigorous mathematical theory began in 2001 in a paper by Bollobás, Riordan, Spencer, Tusnády ($*$)

---

* *The degree sequence of a scale-free random graph process*, RSA 2001

The Barabási-Albert model

The classical version of the model is as follows:

- Start with an initial graph G_0 (finite)
- Obtain G_{n+1} from G_n in the following way:

  add a vertex v_{n+1} to G_n and connect it to a vertex u in G_n selected independently of the past and with probability proportional to its degree

$$P\left(v_{n+1} \text{ connects to } u \mid G_n\right) = \frac{\text{degree of } u \text{ in } G_n}{\text{sum of all degrees in } G_n}$$

This equation is the so-called preferential attachment rule; new vertices are more likely to connect themselves to "popular" vertices (with higher degree)

Barabási & Albert; *Emergence of scaling in random networks*, Science 1999 (...cited by 46102)

The Barabási-Albert model

The classical version of the model is as follows:

- Start with an initial graph G_0 (finite)
- Obtain G_{n+1} from G_n in the following way:

  add a vertex v_{n+1} to G_n and connect it to a vertex u in G_n selected independently of the past and with probability proportional to its degree

$$P\left(v_{n+1} \text{ connects to } u \mid G_n\right) = \frac{\text{degree of } u \text{ in } G_n}{\text{sum of all degrees in } G_n}$$

This equation is the so-called preferential attachment rule; new vertices are more likely to connect themselves to "popular" vertices (with higher degree)

Initial graph: a single vertex with a self-loop (G_0^{loop}) or a single edge (G_0^{edge})

Barabási & Albert; *Emergence of scaling in random networks*, Science 1999 (...cited by 46102)

Some known results for the BA model

Power-law Degree Distribution

$N_{G_n}(d) =$ number of vertices having degree exactly d in G_n

$|G_n| =$ number of vertices in G_n

$$\forall d \geq 1, \qquad \lim_{n\to\infty} \frac{N_{G_n}(d)}{|G_n|} = \frac{4}{d(d+1)(d+2)}, \qquad P_{G_0^{\text{edge}}}\text{-almost surely}$$

Bollobás, Riordan, Spencer, Tusnády; *The degree sequence of a scale-free random graph process*, 2001

Móri; *On random trees*, Studia Sci. Math. Hungar. 2002

Some known results for the BA model

Height [1]

$$\lim_{n\to\infty} \frac{\text{height of } G_n}{\log |G_n|} = \frac{1}{2c}, \qquad P_{G_0^{\text{edge}}}\text{-almost surely}$$

where c is the solution of $ce^{c+1} = 1$

Maximum Degree [2]

$$\lim_{n\to\infty} \frac{\text{max. degree}(G_n)}{\sqrt{|G_n|}} = \zeta, \qquad P_{G_0^{\text{edge}}}\text{-almost surely}$$

where ζ is an a.s. positive and finite random variable (with an absolutely continuous distribution)

---

[1] van der Hofstad; *Random graphs and complex networks, vol. 2*, Cambridge Series in Statistical and Probabilistic Mathematics 2024

[2] Móri; *The maximum degree of the Barabási–Albert random tree*, CPC 2005

Some known results for the BA model

Degrees of k-th level vertices

$X[n,k] =$ number of vertices at level k after step n

$X[n,k,d] =$ number of vertices at level k with degree d, after step n

$$\forall d \geq 1, k \geq 1, \qquad \lim_{n\to\infty} \frac{X[n,k,d]}{X[n,k]} = \frac{1}{d(d+1)}, \qquad P_{G_0^{\text{edge}}}\text{-almost surely}$$

Still follows a power law, the exponent is not 3 but 2, for each level $k \geq 1$

Móri; *A surprising property of the Barabási-Albert random tree*, Studia Sci. Math. Hungar. 2006

Global information in BA model

BA model was originally proposed to explain the power-law degree distribution phenomenon observed in many real networks

BA model is based on two principles:

1) growing nature of the network
   to understand why real networks are as such we have to account for the fact that networks have evolved till reaching the current status

2) preferential attachment mechanism
   rich-get-richer phenomenon

In BA model, in order to describe G_{n+1} from G_n, the degree of all the nodes in G_n must be known: the model is based on a global information

Global information in BA model

A network growth mechanism based on global information is not realistic

To circumvent this problem network growth model based on random walk have been introduced

* *Saramäki-Kaski (2004) – Scale-free networks generated by RWs*

Although local attachment rule, a primitive is required to randomly select networks node (to restart the walker) (not purely local)

We propose and study a network growth model which is purely local

The Tree Builder Random Walk (TBRW)

The TBRW is a stochastic process $\{(T_n, X_n)\}_{n\geq 0}$ where T_n is a tree and X_n, a vertex of the tree, is the current position of a random walker

Parameters: a sequence of probability laws $\mathcal{L} := \{\mathcal{L}_n\}_{n\geq 1}$ and an i.c. (T_0, x_0)

* each $\mathcal{L}_n$ is supported on nonnegative integers
* T_0 is a finite rooted tree with a self-loop at the root (*no periodicity*)
* x_0 is a vertex of T_0 (*initial position of the walker*)

The Tree Builder Random Walk (TBRW)

The TBRW is a stochastic process $\{(T_n, X_n)\}_{n\geq 0}$ where T_n is a tree and X_n, a vertex of the tree, is the current position of a random walker

Parameters: a sequence of probability laws $\mathcal{L} := \{\mathcal{L}_n\}_{n\geq 1}$ and an i.c. (T_0, x_0)

* each $\mathcal{L}_n$ is supported on nonnegative integers
* T_0 is a finite rooted tree with a self-loop at the root (*no periodicity*)
* x_0 is a vertex of T_0 (*initial position of the walker*)

At every time n, a random number of leaves, distributed according to $\mathcal{L}_n$, independent of everything else, are added to X_{n-1}

If a positive number of leaves is drawn, the tree T_{n-1} is modified immediately

After that, the random walk takes a step on the possibly modified tree to a uniformly chosen neighbor, before the next leaves are drawn

The Tree Builder Random Walk (TBRW)

We refer to this model as $\mathcal{L}$-TBRW to emphasize the dependence on the sequence $\mathcal{L} := \{\mathcal{L}_n\}_{n\geq 1}$

The law of $\{(T_n, X_n)\}_{n\geq 0}$ when $(T_0, X_0) = (T_0, x_0)$ is denoted by

$$\mathbb{P}_{T_0,x_0;\mathcal{L}} \qquad (\textit{and the corresponding expectation by } \mathbb{E}_{x_0,T_0;\mathcal{L}})$$

$Z := \{Z_n\}_{n\geq 1}$ a sequence of independent nonnegative integer valued random variables such that $Z_n \sim \mathcal{L}_n$, i.e., Z_n is the number of leaves added at time n

TBRW's chronology

The TBRW model has been introduced and studied under the assumption of *uniform ellipticity* on the sequence of laws $\mathcal{L}$, i.e., $\mathcal{L}_n(\{0\}) \le 1 - \kappa \ \forall n$ in

I., Ribeiro, Valle, Zuaznábar; *Tree builder random walk: Recurrence, transience and ballisticity*, Bernoulli 2022

The TBRW is a generalization of the models studied in

I., Figueiredo, Neglia; *Transient and slim versus recurrent and fat: Random walks and the trees they grow*, JAP 2019

Figueiredo, I., Oliveira, Reed, Ribeiro; *On a random walk that grows its own tree*, EJP 2021

The TBRW has been also studied dropping the uniform ellipticity assumption

Engländer, I., Ribeiro; *Recurrence, transience and degree distribution for the TBRW*, AIHP ???

Two different perspectives of TBRW model

The TBRW model consists of a random walk moving on a network that grows over time and their behaviors are strongly intertwined (mutual dependence)

* network growth depends on how the random walk moves
* random walk possible moves depend on the underlying network

Two different perspectives of TBRW model

The TBRW model consists of a random walk moving on a network that grows over time and their behaviors are strongly intertwined (mutual dependence)

* network growth depends on how the random walk moves
* random walk possible moves depend on the underlying network

TWO PERSPECTIVES:

- network structure
- random walk behavior

In this talk we will focus on the TBRW as a network growth model

Asymptotic Graph Property

A family $\mathcal{G}$ of sequences of (finite) graphs is an asymptotic graph property if satisfying $\mathcal{G}$ does not depend on a finite number of coordinates, i.e.,

$$\forall s \in \mathbb{N}, \qquad \{H_t\}_{t\in\mathbb{N}} \in \mathcal{G} \iff \{H_{t+s}\}_{t\in\mathbb{N}} \in \mathcal{G}$$

Asymptotic Graph Property

A family $\mathcal{G}$ of sequences of (finite) graphs is an asymptotic graph property if satisfying $\mathcal{G}$ does not depend on a finite number of coordinates, i.e.,

$$\forall s \in \mathbb{N}, \qquad \{H_t\}_{t\in\mathbb{N}} \in \mathcal{G} \iff \{H_{t+s}\}_{t\in\mathbb{N}} \in \mathcal{G}$$

A few examples of asymptotic graph properties:

- $\mathcal{G} = \left\{ \{G_n\}_n \,:\, \lim_{n\to\infty} \frac{|G_n|}{n} = 1 \right\}$ (*size linear growth*)
- $\mathcal{G}_h = \left\{ \{G_n\}_n \,:\, \lim_{n\to\infty} \frac{\text{height of } G_n}{\log |G_n|} = h \right\}$ (*height*)
- $\mathcal{G}_h = \left\{ \{G_n\}_n \,:\, \lim_{n\to\infty} \frac{\text{max. degree of } G_n}{|G_n|^{1/2}} > h \right\}$ (*maximum degree*)
- $\mathcal{G}_h := \left\{ \{G_n\}_n \,:\, \lim_{n\to\infty} \frac{N_{G_n}(d)}{|G_n|} = h \right\}$ (*density of vertices with degree d*)

Asymptotic graph holding almost surely in BA model

P_{G_0} = law of the BA-model, conditioned on the initial graph being G_0

A random graph sequence $\{G_t\}_{t\in\mathbb{N}}$ satisfies P_{G_0}-a.s. the graph property $\mathcal{G}$, if

$$P_{G_0}(\{G_t\}_{t\in\mathbb{N}} \in \mathcal{G}) = 1$$

- $\mathcal{G}_h = \left\{ \{G_n\}_n \,:\, \lim_{n\to\infty} \frac{\text{height of } G_n}{\log |G_n|} = h \right\} \implies$ holds P_{G_0}-a.s. for $h = \frac{1}{2c}$
- $\mathcal{G}_d := \left\{ \{G_n\}_n \,:\, \lim_{n\to\infty} \frac{N_{G_n}(d)}{|G_n|} = h \right\}$, holds P_{G_0}-a.s. for $h = \frac{4}{d(d+1)(d+2)}$
- $\mathcal{G}_h = \left\{ \{G_n\}_n \,:\, \lim_{n\to\infty} \frac{\text{max. degree of } G_n}{|G_n|^{1/2}} > h \right\}$

Móri shown that it has a $P_{G_0^{\text{edge}}}$ non-trivial probability for any $h > 0$, hence it is not a $P_{G_0^{\text{edge}}}$-almost sure graph property

Móri; *The maximum degree of the Barabási–Albert random tree*, CPC 2005

Graph sequence generated by the TBRW

Growth times: a sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ corresponding to the times when the TBRW process adds at least one new vertex: $\tau_0 \equiv 0$

$$\tau_k := \inf\{n > \tau_{k-1} \, : \, Z_n \geq 1\}, \quad k \geq 1$$

if $\tau_{k-1} < \infty$, and $\tau_k = \infty$ otherwise

Graph sequence generated by the TBRW

Growth times: a sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ corresponding to the times when the TBRW process adds at least one new vertex: $\tau_0 \equiv 0$

$$\tau_k := \inf\{n > \tau_{k-1} \,:\, Z_n \geq 1\}, \quad k \geq 1$$

if $\tau_{k-1} < \infty$, and $\tau_k = \infty$ otherwise

We consider only sequences of probability laws $\{\mathcal{L}_n\}_{n\geq 1}$ under which $\tau_k < \infty$ holds for all k almost surely (not interested is scenarios in which the graph sequence will eventually stop growing)

The sequence $\{T_{\tau_k}\}_{k\in\mathbb{N}}$ is the subsequence of $\{T_n\}_{n\in\mathbb{N}}$ that carries all modifications made by the walker, i.e., $T_n = T_{\tau_k}$ for all $n \in [\tau_k, \tau_{k+1})$

Main result

We focus on $\mathcal{L}$-TBRW with $\mathcal{L}_n = \text{Ber}(p_n)$

We identify a condition (M) under which we can transfer asymptotic graph properties which hold almost surely for the BA model to the graph sequence generated by the TBRW

Theorem [Transfer Principle]

Let $\mathcal{G}$ be an asymptotic graph property and consider a $\mathcal{L}$-TBRW satisfying condition (M). Then,

$$\underbrace{P_{G_0^{\text{loop}}}\left(\{G_t\}_{t\in\mathbb{N}} \in \mathcal{G}\right) = 1}_{BA} \implies \underbrace{\mathbb{P}_{T_0,x_0;\mathcal{L}}\left(\{T_{\tau_k}\}_{k\in\mathbb{N}} \in \mathcal{G}\right) = 1}_{TBRW}, \textit{ for all } (T_0, x_0)$$

Main Corollaries

Corollary [Power-law degree distribution]

Consider a $\mathcal{L}$-TBRW satisfying condition (M). Then, for any finite initial condition (T, x) and degree $d \geq 1$

$$\lim_{n\to\infty} \frac{N_{T_n}(d)}{|T_n|} = \frac{4}{d(d+1)(d+2)}, \quad \mathbb{P}_{T,x;\mathcal{L}}\text{-almost surely}$$

This was proved in (*) for $\mathcal{L}_n = \mathrm{Ber}(n^{-\gamma})$ with $\gamma \in (2/3, 1]$

Corollary [Height of T_n]

Let c be the solution of the equation $ce^{c+1} = 1$. Then, for any $\mathcal{L}$-TBRW satisfying condition (M), and any finite i.c. (T, x)

$$\lim_{n\to\infty} \frac{\text{height of } T_n}{\log |T_n|} = \frac{1}{2c}, \quad \mathbb{P}_{T,x;\mathcal{L}}\text{-almost surely}$$

(∗) Engländer, I., Ribeiro; *Recurrence, transience and degree distribution for the TBRW*, AIHP ???

Main Corollaries

Corollary [Maximum degree]

Consider an $\mathcal{L}$-TBRW satisfying condition (M). Then, for any finite i.c. (T, x), there exists a strictly positive random variable ζ such that

$$\lim_{n\to\infty} \frac{\text{max degree of } T_n}{\sqrt{|T_n|}} = \zeta, \quad \mathbb{P}_{T,x;\mathcal{L}}\text{-almost surely}$$

Corollary [Degrees of kth level vertices]

Consider an $\mathcal{L}$-TBRW satisfying condition (M). Then, for any finite i.c. (T, x) and for every $k \geq 1,\ d \geq 1$

$$\lim_{n\to\infty} \frac{X[n,k,d]}{X[n,k]} = \frac{1}{d(d+1)}, \quad \mathbb{P}_{T,x;\mathcal{L}}\text{-almost surely}$$

Proof ideas and technical difficulties

The proof follows from a coupling with the BA-model

Main idea: If the sequence $(p_n)_{n\geq 1}$ goes fast enough to zero, the walker mixes before adding new vertices, thus new vertices are added according to the stationary measure

Two main issues to overcome:

a) Good bounds for the mixing time depend on structural properties of the graph, such as its diameter

b) Mixing before adding new vertices ensures that new vertices are added according to a distribution that is “close” to the stationary distribution

   Since we are not exactly in the stationary distribution, how to couple the TBRW with the BA model?

Proof ideas and technical difficulties: challenge a)

Mixing time bound: The easiest upper bound for mixing time is the number of vertices squared (*)

Since the order of the graph at time n is a sum of independent random variables, we have good control over this quantity

However, depending on $(p_n)_{n\geq 1}$, we may observe infinitely often the walker adding a new vertex before taking this many steps

This leads us to obtain a stronger general bound for mixing time on trees

Lemma 1 [Mixing time bound on trees]

For a SSRW on a tree $T = (S, E)$ with a (unique) self-loop, it holds that

$$t_{\text{mix}}(\varepsilon) \leq (2\,\text{diam}(T) + 1)|E| \log\left(\frac{2|E|}{\varepsilon}\right)$$

---

* Brightwell, Winkler - *Extremal cover times for random walks on trees*, J. Graph Theory 1990

Proof ideas and technical difficulties: challenge b)

Strong stationary times: the walker having mixed on a graph does not imply that it is distributed according to the stationary distribution

To couple the TBRW with the BA model, we need that the new leafs are added at a position chosen according to the stationary distribution

This leads us to extend the notion of stationary times to the TBRW

Dealing with stationary times comes with extra challenges as well: we also need to ensure that the random times at which new vertices are added after the stationary times remain stationary times

Strong stationary times for classical (finite) Markov chains

Let $\{Y_t\}_{t\geq 0}$ be a Markov chain with stationary distribution π. A strong stationary time for $\{Y_t\}_t$ and starting point y_0 is a stopping time η such that Y_η is distributed as π and it is independent of η, i.e., $\forall y$ and $k \in \mathbb{N}$

$$P_{y_0}\left(Y_\eta = y, \eta = k\right) = \pi(y) P_{y_0}\left(\eta = k\right)$$

Proposition [Aldous/Diaconis (1987)] *For any finite Markov chain $\{Y_t\}_t$ and any initial state y_0, there exists a strong stationary time η_{y_0} (optimal) s.t.*

$$P_{y_0}(\eta_{y_0} > t) = s_{y_0}(t), \qquad \forall t > 0$$

where, $s_x(t) := \max_{y\in S}\left[1 - \frac{P^t(x,y)}{\pi(y)}\right]$ *is the separation distance from* x

Strong stationary times for classical Markov chains

Lemma [Peres] *For any reversible Markov chain,* $s(2t) \leq 4d(t)$*, where* $d(t) := \max_{x \in S} \|P^t(x, \cdot) - \pi\|_{TV}$ *and* $s(t) := \max_{x \in S} s_x(t)$

Proposition 2 [tail of an optimal strong stationary time and t_{mix}]

Let $\{Y_t\}_t$ *be an aperiodic and irreducible Markov chain and let* η_{x_0} *be an optimal strong stationary time for the chain started at* x_0*. Then*

$$P_{x_0}\left(\eta_{x_0} > \ell t_{mix}\right) \leq 4 \cdot 2^{-\ell/2}, \qquad \forall \ell \in \mathbb{N}$$

Strong stationary times for TBRW

Since in TBRW the tree over which the walker is walking changes over time, what do we mean by strong stationary times in the context of the TBRW?

The walker of the TBRW can be coupled to agree with a SSRW until the appearance of new vertices; after that the two walkers evolve independently

- TBRW starting at (T, x)
- $\widetilde{X}$ be a SSRW on T started at x
- τ be the stopping time when new vertices are added for the first time
- $P_{T,x;\mathcal{L}}$ be the coupling measure of X and $\widetilde{X}$ such that:
  * $X_i = \widetilde{X}_i \qquad$ for $i \leq \tau$
  * $\{X\}_{i \geq \tau}$ and $\{\widetilde{X}\}_{i \geq \tau}$, given τ and the vertex $X_\tau = \widetilde{X}_\tau$, are independent

Strong stationary times for TBRW

Let η_x be a strong stationary time for $\widetilde{X}$, and let π_T be the stationary distribution of $\widetilde{X}$ on T, i.e., $\pi_T(v) \propto \deg_T(v)$

The next result concerns the distribution of X_{η_x} in the coupling $P_{T,x;\mathcal{L}}$

Proposition 3

Let T be a finite tree, x, v two of its vertices, and consider a sequence of laws $\mathcal{L}$ satisfying that $\mathcal{L}_n(\{0\}) > 0$ for all n. Then, conditioned on $\{\eta_x < \tau\}$, both X_{η_x} and $X_{\tau-1}$ follow the distribution π_T, i.e.,

$$P_{T,x;\mathcal{L}}\left(X_{\eta_x} = v \mid \eta_x < \tau\right) = P_{T,x;\mathcal{L}}\left(X_{\tau-1} = v \mid \eta_x < \tau\right) = \pi_T(v)$$

$\mathcal{L}_n(\{0\}) > 0, \forall n$ is to make sure the event $\{\eta_x < \tau\}$ has positive probability

Mixing condition (M)

Mixing condition

We say that $\mathcal{L}$-TBRW satisfies (M) if the following conditions hold:

1) For all n, $\mathcal{L}_n = \mathrm{Ber}(p_n)$, with $p_n \in (0,1)$
2) For any finite tree initial condition (T,x), there exists a sequence of optimal strong stationary times $\{\eta_k\}_k$ such that

$$P_{T,x;\mathcal{L}}\left(\eta_k > \Delta\tau_k \; i.o.\right) = 0$$

Mixing condition (M)

1) For all n, $\mathcal{L}_n = \text{Ber}(p_n)$, with $p_n \in (0,1)$

At each step the walker can add at most one vertex, and has a positive chance both for adding one and for adding none. This is necessary for two reasons:

* $\mathcal{L}_n(\{1\}) = 1$, it was proved that the TBRW limiting random tree is one-ended, which is not true for the BA model$(*)$

* $\mathcal{L}_n(\{1\}) = 0$ the walker is just a SSRW over the initial condition

---

* Figueiredo, I., Imbuzeiro, Reed, Ribeiro - *On a random walk that grows its own tree*, EJP 2021

Mixing condition (M)

1) For all n, $\mathcal{L}_n = \text{Ber}(p_n)$, with $p_n \in (0,1)$

At each step the walker can add at most one vertex, and has a positive chance both for adding one and for adding none. This is necessary for two reasons:

* $\mathcal{L}_n(\{1\}) = 1$, it was proved that the TBRW limiting random tree is one-ended, which is not true for the BA model$(*)$
* $\mathcal{L}_n(\{1\}) = 0$ the walker is just a SSRW over the initial condition

2) For any finite initial condition (T, x), there exists a sequence of optimal strong stationary times $\{\eta_k\}_k$ such that

$$P_{T,x;\mathcal{L}}\left(\eta_k > \Delta\tau_k \text{ i.o.}\right) = 0$$

This guarantees that after some random time, all subsequent new vertices are added after the walker has mixed on the current tree

* Figueiredo, I., Imbuzeiro, Reed, Ribeiro - *On a random walk that grows its own tree*, EJP 2021

Coupling TBRW and BA model

Theorem

Consider a TBRW satisfying condition (M). Then, there exists a probability space containing the TBRW and a sequence of random graph processes $\{\mathbf{G}_n\}_{n\in\mathbb{N}}$*, where* $\mathbf{G}_n := \{G_k^{(n)}\}_{k\in\mathbb{N}}$ *is a BA-tree starting at* T_{τ_n}*, s.t.*

$$\lim_{n\to\infty} D_{TV}\left(\{T_{\tau_{n+k}}\}_{k\in\mathbb{N}}, \{G_k^{(n)}\}_{k\in\mathbb{N}}\right) = 0$$

In words, we can construct a probability space containing a TBRW satisfying (M) and a sequence of BA-trees with the property that the processes are getting closer, in total variation distance

Coupling TBRW and BA model (proof sketch)

We will have a single TBRW process $\{(T_k, X_k)\}_{k\geq 0}$, with growth times $(\tau_n)_n$

We also use a sequence $(\eta_n)_{n\geq 0}$ of optimal strong stationary times for SSRW random walk on T_{τ_n} started at X_{τ_n}, independently for different n's

For each n, let $\{U_k^{(n)}\}_{k\geq 1}$ be a sequence of i.i.d. $\mathrm{Uni}[0,1]$ random variables independent of the TBRW for all n and k. We generate $G_1^{(n)}$ as follows:

(A) If $\eta_n < \tau_{n+1} - \tau_n =: \Delta\tau_n$, then we set $G_1^{(n)} = T_{\tau_{n+1}}$

(B) Otherwise, we use $U_1^{(n)}$, and independently of everything else, select a vertex of $v \in T_{\tau_n}$ with probability $\frac{\deg_{T_{\tau_n}}(v)}{\sum_{z\in T_{\tau_n}} \deg_{T_{\tau_n}}(z)}$ and connect it to a new vertex added to $G_0^{(n)} := T_{\tau_n}$. This new graph is then $G_1^{(n)}$

Coupling TBRW and BA model (proof sketch)

Option (A) is the “useful one”: if that occurs, then we will be able to continue the coupling by generating $G_2^{(n)}$ starting from $G_1^{(n)} = T_{\tau_{n+1}}$

If (B) occurs, then we cannot relate $G_1^{(n)}$ to $T_{\tau_{n+1}}$, hence we consider the coupling to be a failure, and will just generate the futures independently

Coupling TBRW and BA model (proof sketch)

Option (A) is the "useful one": if that occurs, then we will be able to continue the coupling by generating $G_2^{(n)}$ starting from $G_1^{(n)} = T_{\tau_{n+1}}$

If (B) occurs, then we cannot relate $G_1^{(n)}$ to $T_{\tau_{n+1}}$, hence we consider the coupling to be a failure, and will just generate the futures independently

$G_1^{(n)}$ is distributed as the first step of a BA graph started from $G_0^{(n)} = T_{\tau_n}$

A graph H is called admissible for T_{τ_n} if H is T_{τ_n} together with a new vertex v_* connected to some vertex u of T_{τ_n}

$$P_{T_0,x_0;\mathcal{L}}\left(G_1^{(n)} = H \middle| \mathcal{F}_{\tau_n}\right) = \underbrace{P_{T_0,x_0;\mathcal{L}}\left(G_1^{(n)} = H, \eta_n < \Delta\tau_n \middle| \mathcal{F}_{\tau_n}\right)}_{(a)} + \underbrace{P_{T_0,x_0;\mathcal{L}}\left(G_1^{(n)} = H, \eta_n \geq \Delta\tau_n \middle| \mathcal{F}_{\tau_n}\right)}_{(b)}$$

Coupling TBRW and BA model (proof sketch)

When $\eta_n \geq \Delta\tau_n$, we generate $G_1^{(n)}$ using $U_1^{(n)}$

$$(b) = \frac{\deg_{T_{\tau_n}}(u)}{\sum_{z \in T_{\tau_n}} \deg_{T_{\tau_n}}(z)} P_{T_0,x_0;\mathcal{L}}\left(\eta_n \geq \Delta\tau_n | \mathcal{F}_{\tau_n}\right)$$

When $\eta_n < \Delta\tau_n$, the vertex v_* added at time τ_{n+1} connects to u if and only if, $X_{\tau_{n+1}-1} = u$. Thus, by the strong Markov property,

$$\begin{aligned}
(a) &= P_{T_0,x_0;\mathcal{L}}\left(X_{\tau_{n+1}-1} = u, \eta_n < \Delta\tau_n \middle| \mathcal{F}_{\tau_n}\right) \\
&= P_{T_{\tau_n},X_{\tau_n};\mathcal{L}^{(\tau_n)}}\left(X_{\tau_1-1} = u, \eta_0 < \tau_1\right) \\
(Prop.3) &= \frac{\deg_{T_{\tau_n}}(u)}{\sum_{z \in T_{\tau_n}} \deg_{T_{\tau_n}}(z)} P_{T_{\tau_n},X_{\tau_n};\mathcal{L}^{(\tau_n)}}\left(\eta_0 < \tau_1\right) \\
&= \frac{\deg_{T_{\tau_n}}(u)}{\sum_{z \in T_{\tau_n}} \deg_{T_{\tau_n}}(z)} P_{T_0,x_0;\mathcal{L}}\left(\eta_n < \Delta\tau_n | \mathcal{F}_{\tau_n}\right)
\end{aligned}$$

Under $P_{T_0,x_0;\mathcal{L}}$, $G_1^{(n)}$ is distributed as one step of a BA (started from T_{τ_n})

Coupling TBRW and BA model (proof sketch)

We construct $\{G_k^{(n)}\}_{k\in\mathbb{N}}$ distributed as a BA starting from $G_0^{(n)} = T_{\tau_n}$, inductively in k (using the strong Markov property of $\{(T_n, X_n)\}_n$)

Assume we have successfully constructed $\{G_i^{(n)}\}_{i=0}^k$, which represents k steps of the BA model with $G_0^{(n)} = T_{\tau_n}$. Then, we generate $G_{k+1}^{(n)}$ as follows:

(A) If $\eta_{n+i} < \Delta\tau_{n+i}$, for all $i \le k$, then we set $G_{k+1}^{(n)} = T_{\tau_{n+k+1}}$

(B) Otherwise, we use $U_{k+1}^{(n)}$, and independently of everything else select a vertex of $v \in G_k^{(n)}$ with probability $\frac{\deg_{G_k^{(n)}}(v)}{\sum_{z\in G_k^{(n)}} \deg_{G_k^{(n)}}(z)}$ and connect it to a new vertex added to $G_k^{(n)}$. This new graph is then $G_{k+1}^{(n)}$

Coupling TBRW and BA model (proof sketch)

Let us consider the following event

$$A_k^{(n)} = \bigcap_{i=0}^{k} \{\eta_{n+i} < \Delta\tau_{n+i}\}$$

Now notice that for any n, we have the following inclusion of events

$$\bigcap_{i=1}^{\infty} \{\eta_{n+i} < \Delta\tau_{n+i}\} \subset \left\{T_{\tau_{n+i}} = G_i^{(n)}, \forall i\right\}$$

which implies that

$$D_{TV}(\{T_{\tau_{n+k}}\}_{k\in\mathbb{N}}, \{G_k^{(n)}\}_{k\in\mathbb{N}}) \leq P_{T_0,x_0;\mathcal{L}}\left(\bigcup_{i=1}^{\infty} \{\eta_{n+i} > \Delta\tau_{n+i}\}\right)$$

By condition (M) we have that

$$\lim_{n\to\infty} D_{TV}\big(\{T_{\tau_{k+n}}\}_{k\in\mathbb{N}}, \{G_k^{(n)}\}_{k\in\mathbb{N}}\big) \leq P_{T,x;\mathcal{L}}\left(\eta_k > \Delta\tau_k\,,\ i.o.\right) = 0$$

Proof sketch of the Trasfer Principle

Theorem [Transfer Principle]

Let $\mathcal{G}$ be an asymptotic graph property and consider a $\mathcal{L}$-TBRW satisfying condition (M). Then,

$$P_{G_0^{\mathrm{loop}}}\left(\{G_t\}_{t\in\mathbb{N}} \in \mathcal{G}\right) = 1 \implies \mathbb{P}_{T_0,x_0;\mathcal{L}}\left(\{T_{\tau_k}\}_{k\in\mathbb{N}} \in \mathcal{G}\right) = 1, \textit{ for all } (T_0, x_0)$$

Let N be the following random variable

$$N := \inf\{k \,:\, \eta_n < \Delta\tau_n \text{ for all } n \geq k\}$$

where η_n is the stationary time for the TBRW started at (T_{τ_n}, X_{τ_n}) with sequence of laws $\mathcal{L}^{(\tau_n)}$

By item (2) of condition (M), the random variable N is finite almost surely

On the event $\{N = j\}$, it holds that $T_{\tau_{j+k}} = G_k^{(j)}$, for all $k \in \mathbb{N}$

Proof sketch of the Trasfer Principle

Let $\mathcal{G}$ be an asymptotic graph property for the BA-tree that holds $P_{G_0^{\text{loop}}}$-a.s.

Being an asymptotic property holding $P_{G_0^{\text{loop}}}$-a.s., we have that if $Q_{T_{\tau_j}}$ denotes the distribution of $\{G_k^{(j)}\}_{k\in\mathbb{N}}$, then

$$Q_{T_{\tau_j}}\left(\{G_k^{(j)}\}_{k\in\mathbb{N}} \in \mathcal{G}\right) \equiv 1, \qquad P_{T,x;\mathcal{L}}\text{-a.s.}$$

and, for all $j \in \mathbb{N}$

$$P_{T,x;\mathcal{L}}\left(\{T_{\tau_{j+k}}\}_k \in \mathcal{G}, N=j\right) = P_{T,x;\mathcal{L}}\left(\{G_k^{(j)}\}_k \in \mathcal{G}, N=j\right) = P_{T,x;\mathcal{L}}\left(N=j\right)$$

where we used the fact that the event $\{\{G_k^{(j)}\}_k \in \mathcal{G}\}$ has total probability

Since $\mathcal{G}$ is an asymptotic graph property, we have that

$$P_{T,x;\mathcal{L}}\left(\{T_{\tau_k}\}_k \in \mathcal{G}, N=j\right) = P_{T,x;\mathcal{L}}\left(\{T_{\tau_{j+k}}\}_k \in \mathcal{G}, N=j\right) = P_{T,x;\mathcal{L}}\left(N=j\right)$$

Summing over j and recalling that N is finite almost surely gives the result

A class of TBRW satisfying condition (M)

Theorem

The $\mathcal{L}$-TBRW with $\mathcal{L}_n = \mathrm{Ber}(n^{-\gamma})$ satisfies condition (M) for all $\gamma \in (2/3, 1]$

Some auxiliary results

Lemma 4 [Upper bound on the diameter]

Consider an $\mathcal{L}$-TRBR where $\mathcal{L}_n = \mathrm{Ber}(n^{-\gamma})$ with $\gamma \in (1/2, 1)$. Then, for any $\varepsilon > 0$, $M > 0$, and and initial finite tree T, there exist positive constants C and C' depending on ε, M, γ and $|T|$, such that for all $n \geq 1$

$$\mathbb{P}_{T,x;\mathcal{L}}\left(\mathrm{diam}(T_n) \geq Cn^{\varepsilon}\right) \leq \frac{C'}{n^M}$$

Lemma 5 [Growth events do not occur too early]

Consider a TBRW starting at (T, x) with sequence of laws $\mathcal{L}_n \sim \mathrm{Ber}(n^{-\gamma})$ with $\gamma \in (2/3, 1)$. Let τ_k be the growth times. Then, for any $\varepsilon < \frac{\gamma}{1-\gamma} - 2$ there exists a $\delta'(\varepsilon, \gamma) > 0$

$$\mathbb{P}_{T,x;\mathcal{L}}\left(\Delta\tau_n \leq n^{1+\varepsilon}\right) \leq \frac{1}{n^{1+\delta'}}$$

Ber($n^{-\gamma}$)-TBRW satisfies condition (M) for $\gamma \in (2/3, 1)$

We need to show that $P_{T,x;\mathcal{L}}(\eta_k > \Delta\tau_k, \textit{ i.o.}) = 0$ for any finite i.c. (T, x)

For a fixed k and sufficiently small $\varepsilon > 0$, Lemma 5 yields

$$
\begin{aligned}
P_{T,x;\mathcal{L}}(\eta_k > \Delta\tau_k) &\le P_{T,x;\mathcal{L}}\left(\eta_k > k^{1+\varepsilon}\right) + P_{T,x;\mathcal{L}}\left(\Delta\tau_k \le k^{1+\varepsilon}\right) \\
&\le \underbrace{P_{T,x;\mathcal{L}}\left(\eta_k > k^{1+\varepsilon}\right)}_{(A)} + \frac{1}{k^{1+\delta'}} \quad \textit{for some positive } \delta'
\end{aligned}
$$

$$
(A) \le P_{T,x;\mathcal{L}}\left(\eta_k > k^{1+\varepsilon}, \operatorname{diam}(T_{\tau_{k-1}}) < k^{\frac{\varepsilon}{2}}\right) + P_{T,x;\mathcal{L}}\left(\operatorname{diam}(T_{\tau_{k-1}}) > k^{\frac{\varepsilon}{2}}\right)
$$

On the event $\{\operatorname{diam}(T_{\tau_{k-1}}) < k^{\varepsilon/2}\}$, η_k is the optimal strong stationary time of a random walk on a tree with $k-1$ vertices and diameter smaller than $k^{\varepsilon/2}$ and by Lemma 1, this tree has mixing time at most $k^{1+\varepsilon/2+o(1)}$

Ber($n^{-\gamma}$)-TBRW satisfies condition (M) for $\gamma \in (2/3, 1)$

By the strong Markov property and Proposition 2, it follows that there exist two universal constants C and C' such that

$$P_{T,x;\mathcal{L}}\left(\eta_k > k^{1+\varepsilon}, \mathrm{diam}(T_{\tau_{k-1}}) < k^{\frac{\varepsilon}{2}}\right) \leq P_{T,x;\mathcal{L}}\left(\eta_k > k^{\frac{\varepsilon}{2}}\, t_{\mathrm{mix}}\right) \leq Ce^{-C'k^{\varepsilon/3}}$$

As regards the second term

$$\begin{aligned} P_{T,x;\mathcal{L}}\left(\mathrm{diam}(T_{\tau_{k-1}}) > k^{\frac{\varepsilon}{2}}\right) \leq & P_{T,x;\mathcal{L}}\left(\mathrm{diam}(T_{\tau_{k-1}}) > k^{\varepsilon/2}, \tau_{k-1} < k^{\frac{1+\delta}{1-\gamma}}\right) \\ & + P_{T,x;\mathcal{L}}\left(\tau_{k-1} \geq k^{\frac{1+\delta}{1-\gamma}}\right) \end{aligned}$$

A Chernoff-type bound says that there exists a constant $C = C(\gamma, \delta) > 0$ s.t.

$$\mathbb{P}_{T,x;\mathcal{L}}\left(\tau_{k-1} \geq k^{\frac{1+\delta}{1-\gamma}}\right) = \mathbb{P}_{T,x;\mathcal{L}}\left(\sum_{i=1}^{k^{\frac{1+\delta}{1-\gamma}}} Z_i \leq k-1\right) \leq e^{-Ck^{1+\delta}}$$

Ber($n^{-\gamma}$)-TBRW satisfies condition (M) for $\gamma \in (2/3, 1)$

Using that the diameter is non-decreasing, Lemma 4 gives us that for large enough k the following holds:

$$\mathbb{P}_{T,x;\mathcal{L}}\left(\operatorname{diam}(T_{\tau_{k-1}}) > k^{\varepsilon/2}, \tau_{k-1} < k^{\frac{1+\delta}{1-\gamma}}\right) \leq \mathbb{P}_{T,x;\mathcal{L}}\left(\operatorname{diam}\left(T_{k^{\frac{1+\delta}{1-\gamma}}}\right) > k^{\varepsilon/2}\right) \leq \frac{1}{k^2}$$

Overall we obtain that

$$P_{T,x;\mathcal{L}}\left(\eta_k > \Delta\tau_k\right) \leq k^{-(1+\delta')} + Ce^{-C'k^{\varepsilon/3}} + k^{-2} + e^{-Ck^{1+\delta}}$$

The claim then follows by the Borel-Cantelli lemma

What about other regimes for TBRW with $\mathcal{L}_n = \text{Ber}(n^{-\gamma})$

Random walk behaviour in TBRW

$\gamma > 1/2$: It was proved in (*) that the random walk in the TBRW is recurrent

$\gamma < 1/2$: We believe that the random walk is transient

$\gamma = 1/2$: No clue about recurrence vs. transience!

Network structure of TBRW

$\gamma > 1/2$: We expect the power-law to hold even for $\gamma \in (1/2, 2/3)$

$\gamma \leq 1/2$: We do not have a guess for the degree distribution

---

$(*)$ Engländer, I., Ribeiro; *Recurrence, transience and degree distribution for the TBRW*, AIHP ???