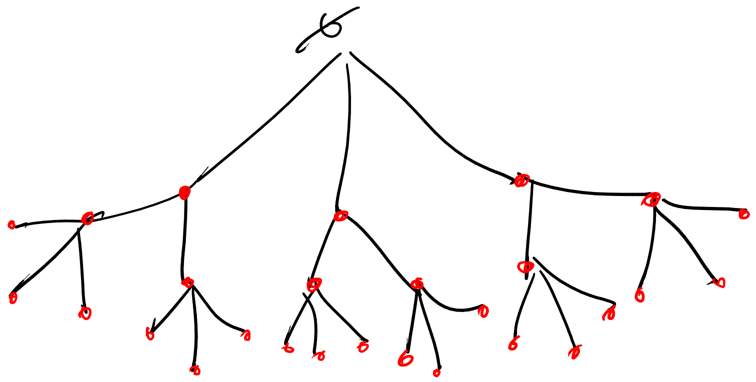
Bouchaud & Dean trap model on Parisi's tree



(Phenomenological model for a (mean field) spin glass dynamics)

Site in generation $i-1$ $\to$ M_i descendants $(i=1,\dots,k)$

Each site $\underline{x_i} = x_1 \dots x_i$ $\to$ Poisson clock $\left(\lambda^i_{\underline{x_i}} = (\sigma^i_{\underline{x_i}})^{-1}\right)$

$\sigma^i = \{\sigma^i_{\underline{x_i}}\}_{\underline{x_i}}$ iid attracted to α_i - stable law, $\alpha_i \in (0,1)$

$\sigma^1, \dots, \sigma^k$ independent ... random env.

Dynamics (on leaves of tree)

When at $\underline{x_k}$, an ascendant $\underline{x_{i-1}}$ of $\underline{x_k}$ is chosen as the mother of the site $(\underline{x_i})$ whose clock rings 1st, and the dynamics jumps from $\underline{x_k}$ to a descendant $\underline{x'_k}$ of $\underline{x_{i-1}}$ in the k-th gen. chosen unif'ly at random.

Alternatively

$\lambda^1_{x_1} + \lambda^2_{x_2} + \dots + \lambda^k_{x_k}$

When at x_k, wait for $\exp(\Lambda^k_{x_k})$; then choose ascendant x_i w.p. $\lambda^i_{x_i}/\Lambda^k_{x_k}$; then jump to descendant of x_{i-1} chosen unif'ly at random in k-th gen.

(Comparison to) Sasaki-Nemoto (GREM-like) trap model

When at x_k, wait for $\exp(\lambda^k_{x_k})$; then choose ascendant x_{i-1} by flipping coins successively on $x_{k-1}, \dots, x_1, \emptyset$, w.p. $p^{k-1}_{x_{k-1}}, \dots, p^1_{x_1}, p^0_0 = 0$ of heads, resp., until getting tails,

$$p^i_{x_i} = \frac{1}{1+(\lambda^i_{x_i})^{-1}} \; ;$$

(x_{i-1} is where succession stops) then choose descendant of x_{i-1} in k-th gen, as before.

Medidas de equilíbrio ($k=2$, pra simplicidade)

$$g(x_2) = \frac{\tau^1_{x_1}}{\sum_{y_1} \tau^1_{y_1}} \; \frac{\dfrac{\tau^2_{x_2}}{\tau^1_{x_1} + \tau^2_{x_2}}}{\sum_{y_2} \dfrac{\tau^2_{x_1 y_2}}{\tau^1_{x_1} + \tau^2_{x_1 y_2}}} \qquad \text{(BDTM)}$$

$$g'(x_2) = \frac{(1+\tau^1_{x_1})\,\tau^2_{x_2}}{\sum_{y_2} (1+\tau^1_{y_1})\,\tau^2_{y_2}} \qquad \text{(SNTM)}$$

Limite de escala (equilíbrio)

Reescalando apropriad/te qdo $m_i \nearrow \infty \ \forall i$:*

$$\mathcal{G}(\underline{x}_2) = \frac{\gamma^1_{x_1}}{\sum_{y_1}\gamma^1_{y_1}} \; \frac{\dfrac{\gamma^2_{x_1x_2}}{\gamma^1_{x_1}+\gamma^2_{x_1,x_2}}}{\sum_{y_2}\dfrac{\gamma^2_{x_1y_2}}{\gamma^1_{x_1}+\gamma^2_{x_1y_2}}} \qquad \text{(BDTM)}$$

$$\mathcal{G}'(\underline{x}_2) = \frac{\gamma^1_{x_1}\gamma^2_{x_1,x_2}}{\sum_{y_1y_2}\gamma^1_{y_1}\gamma^2_{y_1y_2}}, \qquad \text{(SNTM)}$$

→ coincides with ∞-vol. GREM Gibbs measure

where $\{\gamma^1_x\}_{x\in[0,1]}$ increments of α_1-stable subordinator

$\forall x \quad \{\gamma^2_{xy}\}_{y\in[0,1]}$ - - - - - - α_2 - - - - - -

→ all subordinators ind. from each other

For SNTM: $\underline{\alpha_1 < \alpha_2}$

↓

no such need in BDTM

↓

but the authors do not seem to have realized it!

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* com relação entre m_i's apropriada

Want to take the scaling limit of dynamics

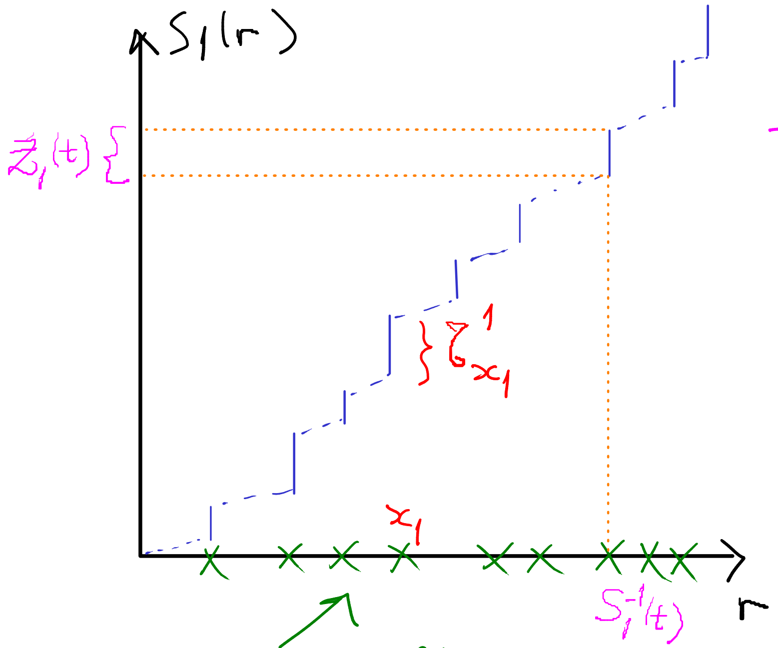
(3 ≠ regimes: near equilibrium; 2 reg's far from eq.) aging

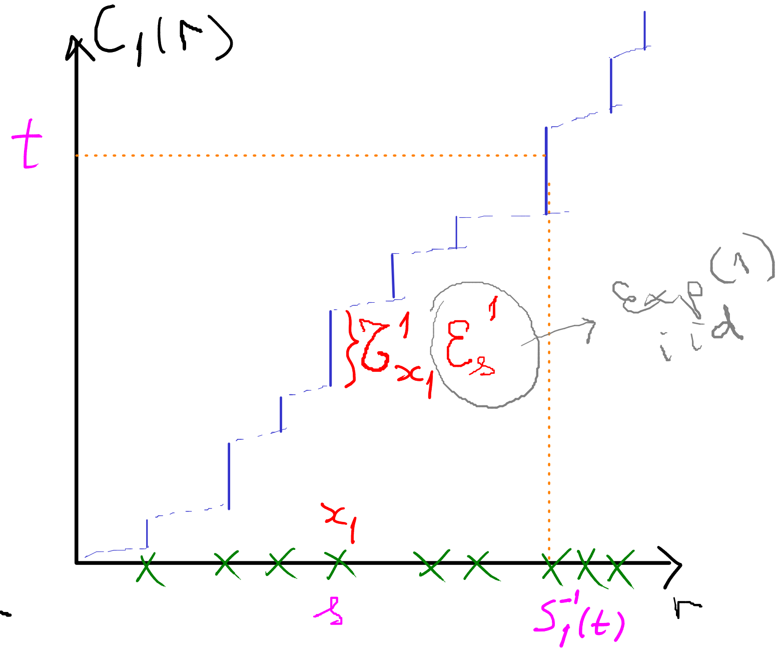
Convenient representation of dynamics:

instead of recording the positions (at given times):
record the values of respective $\tau_{\underline{x}^i}$'s;

think of following construction:

1st level ($k=1$)





some (labeled) marks of superposition of M_1 PPP(1)

→ jumps of $S_1 = \mathcal{J}_1$

clock process

$$C_1(r) = \sum_{\substack{s\in\mathcal{J}_1\\ s\le r}} \xi^1_s\, \mathcal{E}^1_s$$

$$= \int_0^r \mathcal{E}^1_s\, dS_1(s)$$

$$S_1(r) = \sum_{\substack{s\in\mathcal{J}_1\\ s\le r}} \underbrace{\xi^1_s}_{\tau^1_{\sigma_s}}$$

time lag clock process

σ_s → rótulo de s

Limite de escala de $S_1 \Rightarrow$ Limite de escala de C_1

$\Rightarrow$ Limite de escala de Z_1

$$S_1(r) = \sum_{x_1=1}^{M_1 = n} \tau^1_{x_1} \underbrace{N^1_{x_1}(r)} \quad PP(1) \text{ iid}$$

3 regimes de escala

inors. of α_1-stable subordinator

$$1) \quad S_1^{(n)}(r) = \frac{1}{n^{1/\alpha_1}} S_1(r) \xrightarrow[n\to\infty]{d} \sum_{x_1 \in [0,1]} \gamma^1_{x_1} N^1_{x_1}(r)$$

(ergodic regime)

$$= \sum_{s \leq r} \tilde{\xi}^1_s$$

$$Z_1^{(n)}(t) = \frac{1}{n^{1/\alpha_1}} Z_1(n^{1/\alpha_1} t) \xrightarrow[n\to\infty]{d} \tilde{Z}_1(t) \ \ldots \ K\text{-process}$$

$\to$ ergodic

$\Rightarrow$ no aging

$$2) \quad \hat{S}_1^{(n)}(r) = \frac{1}{n^\beta} S_1(a_1^{(n)} r) \ ; \ 0<\beta<1/\alpha_1, \ a_1^{(n)} = n^{\alpha_1 \beta - 1}$$

(poly. sub ergodic regime) $\xrightarrow[n\to\infty]{fc} \hat{S}(r) \ \ldots \ \alpha_1\text{-stable subordinator}$

$$\hat{Z}_1^{(n)}(t) = \frac{1}{n^\beta} Z_1(n^\beta t) \xrightarrow[n\to\infty]{d} \hat{Z}_1(t) \text{ for a.e. } (\tau^1_{x_1})$$

... visits jumps of $\hat{S}$ in order

spends $\mathrm{Exp}(\gamma^1_s)$ in s

$\to$ aging

Aging of $\hat{Z}_1$

$\int_0^\cdot \varepsilon_s^1 \, d\hat{S}_1^{(n)}(s)$

$$P\left(\text{no jump of } \hat{Z}_1^{(n)} \text{ during } (t_w, t_w+t)\right)$$

α_1-stable subordinator

$$= P\left((t_w, t_w+t) \cap \text{Range of } \hat{C}_1^{(n)} = \emptyset\right)$$

$\int_0^\cdot \varepsilon_s^1 \, d\hat{S}_1(s)$

$$\xrightarrow[n\to\infty]{\text{sc}} P\left(\text{-------} \ \hat{C}_1 \text{---}\right)$$

$$= P\left(\text{no jump of } \hat{Z}_1 \text{ during } (t_w, t_w+t)\right) = \underbrace{\text{fn.}}_{} \text{ of } t/t_w$$

arcsine law

$\mathrm{Beta}(1-\alpha_1, \alpha_1) \sim$ $= P\left(U_1 > \frac{t}{t+t_w}\right)$

$\tilde{N}_r^1 \to PP(1)$

3) $\check{S}_1^{(n)}(r) = S_1^{(n)}\left(\frac{r}{n}\right) \xrightarrow[n\to\infty]{d} \sum_{x_1=1}^{\tilde{N}_r^1} \tau_{x_1}^1$ for a.e. $(\tau_{x_1}^1)$

$\overset{!!}{=} \check{S}_1(r)$

(order 1 subergodic regime) $\check{Z}_1^{(n)}(t) \xrightarrow[n\to\infty]{} \check{Z}_1(t)$ for a.e. $(\tau_{x_1}^1)$

$$\bar{S}_1^{(m)}(r) = \frac{1}{m^{1/\alpha_1}} \check{S}_1(mr) \xrightarrow[m\to\infty]{d} \hat{S}_1(r) \ldots \alpha_1\text{-stable subordinator}$$

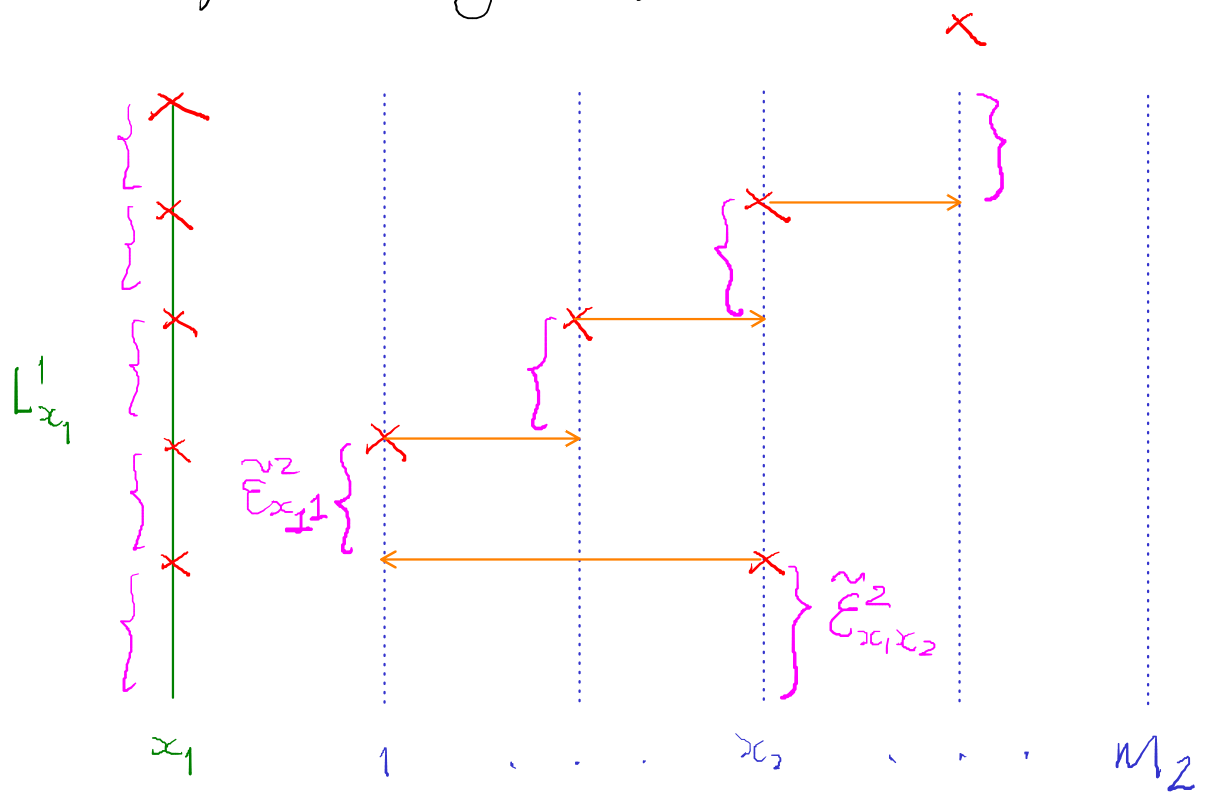
$$\bar{Z}_1^{(m)}(t) = \frac{1}{m^{1/\alpha_1}} \check{Z}_1(mt) \xrightarrow[m\to\infty]{d} \hat{Z}_1(t)$$

Aging: $\lim_{m\to\infty} \lim_{n\to\infty} P\left(\text{no jump of } Z_1 \text{ during } m^{1/\alpha_1}(t_w, t_w+t)\right) = \text{arcsine law}$ for a.e. $(\tau_{x_1}^1)$

2nd Level

Enough to understand dynamics on 2nd level while resting at any given x_1 on the 1st level (during a time interval of length $L^1_{x_1} = \mathrm{Exp}(\lambda^1_{x_1})$).

Key ingredient: durations of visits to level 2 sites before reaching $L^1_{x_1}$

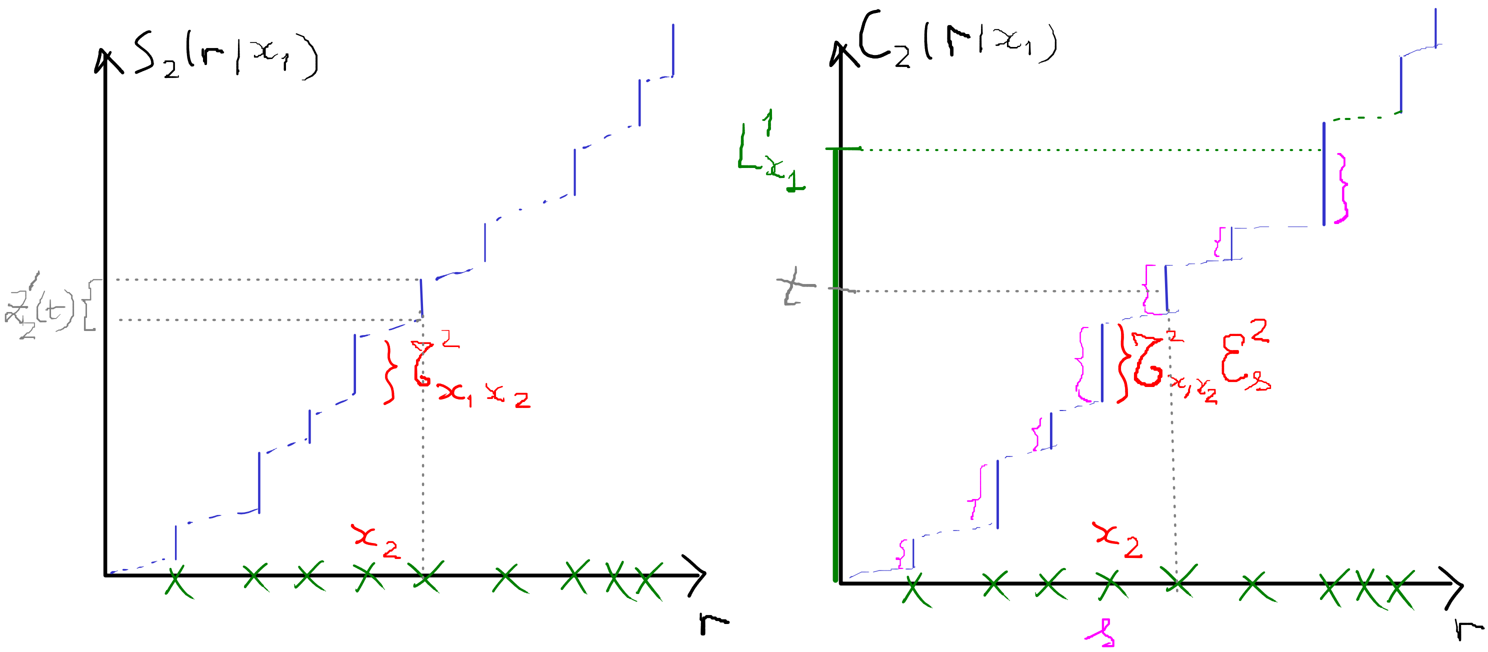


Successive $\tilde{\mathcal{E}}^2_{x_1 x_2}$, $\tilde{\mathcal{E}}^2_{x_1 x_2'}$, ... up to last one (before jumping on 1st level):

ind. exp. r.v.'s w. rates $\lambda^1_{x_1} + \lambda^2_{x_1 x_2}$, $\lambda^1_{x_1} + \lambda^2_{x_1 x_2'}$, ..., resp.

2nd level dynamics during a sojourn of 1st level on x_1:

version of 1st level dynamics:



A more convenient* representation of 2nd level states (instead of $Z_2'(t)$ = current value of $\tau^2_{x_1 x_2}$):

$$Z_2(t) = \text{current value of } \frac{1}{\lambda^1_{x_1} + \lambda^2_{x_1 x_2}}$$

... expected jump time when at $x_1 x_2$ at time t

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* for technical reasons, but it is more natural in the aging investigation context

Scaling limit of 2 level dynamics

(Assume appropriate rel. relation: $M_1^{1/\alpha_1} = M_2^{1/\alpha_2}$;
so that (rescaled) S_1 & $S_2(x_1)$, $x_1 = 1, 2, \ldots$,
converge to the same kind of limit in
regimes 1 & 2 above; regime 3 does not require that.

$\leadsto$ volume construction of 2nd level
dynamics (within sojourns of 1st
level dyn.) as above.

Scaling limit results for dynamics
follow in 3 regimes $\to$ ergodic - no aging
$\to$ poly. sub-erg. $\Big\}$ aging
$\to$ O(1) " "

O(1) aging result

$$\lim_{n\to\infty} \lim_{M_1, M_2 \nearrow \infty} P\left(\text{no jump of } (Z_1, Z_2) \text{ in } n^{1/\alpha_1}[t_w, t_w + t]\right)$$

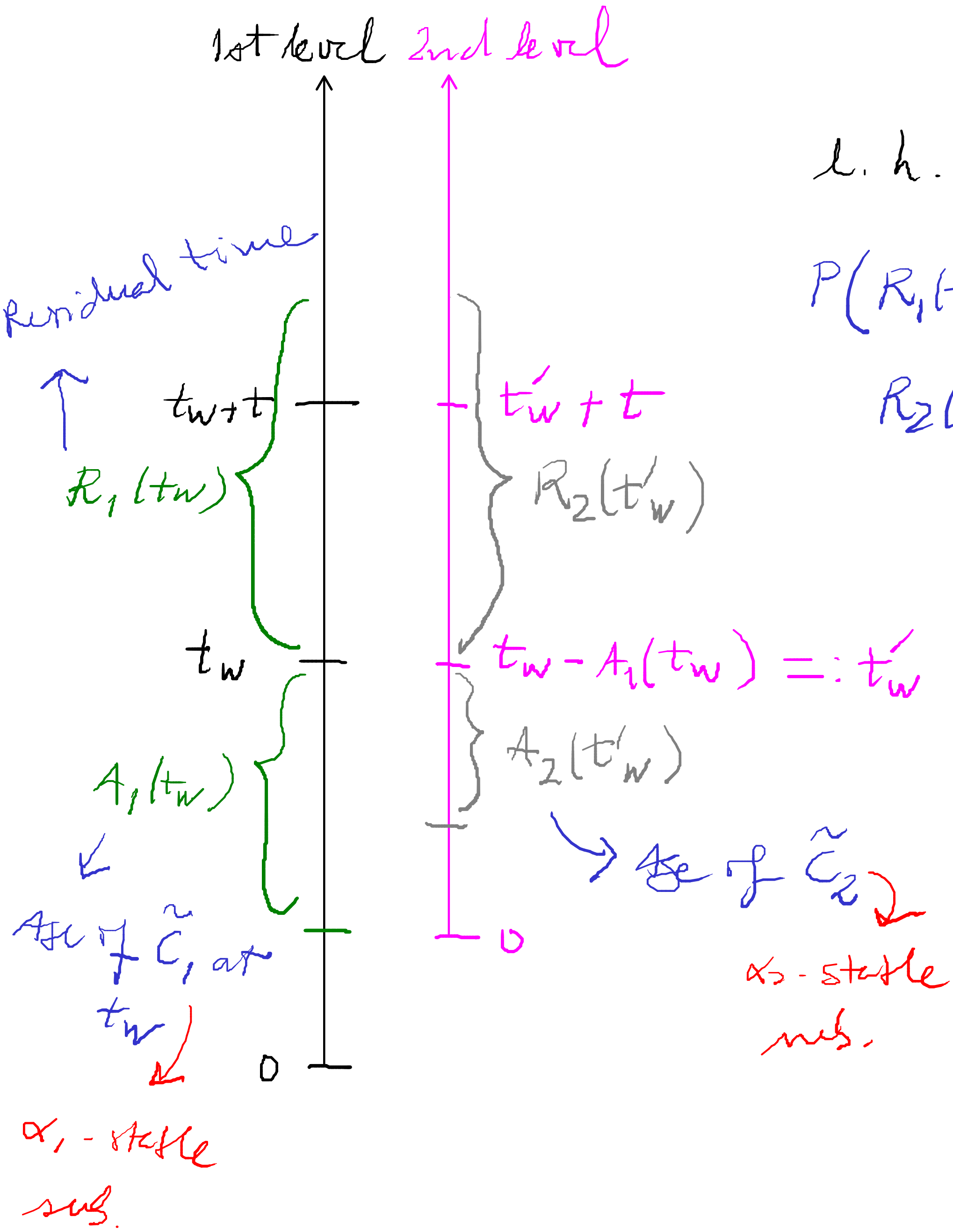
limiting process

$$= P\left(\text{no jump of } (\tilde{Z}_1, \tilde{Z}_2) \text{ in } [t_w, t_w + t]\right)$$

$$\overset{*}{=} P\left(U_1 U_2 > t/(t+t_w)\right) \quad \text{for a.e. } (\sigma_i)$$

$U_i \sim \text{Beta}(1-\alpha_i; \alpha_i)$ indep.

Argument for $*$:



l.h.s. of $*$ =

$$P\big(R_1(t_w) > t_w + t,\ R_2(t_w') > t_w' + t\big)$$

r.h.s. of $*$ follows readily from scaling relations for stable subord. (of index < 1), other known results for these processes & indep.