Resultados Assintóticos em algumas classes de passeios aleatórios

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Random walks on $\mathbb{Z}^d$.

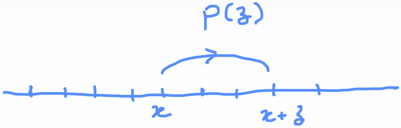
$(\xi_j)_{j\geq 1}$ non zero IID random variables with values in $\mathbb{Z}^d$.

The associated **random walk** with initial condition $X_0 = x$ is

$$X_n = x + \sum_{j=1}^{n} \xi_j \,, \quad n \geq 1.$$

The distribution of the process $(X_n)_{n\geq 0}$ is determined by x and by the **transition probability function**

$$p(z) = P(\xi_1 = z), \; z \in \mathbb{Z}^d.$$



We call the random walk **simple** when

$$\sum_{j=1}^{d}(p(e_j) + p(-e_j)) = 1$$

$(e_j)_{1 \leq j \leq d}$ is the canonical basis $\mathbb{Z}^d$.

Generalization: Random walks on Graphs, collections of interacting random walks.

The random walk is **symmetric** if

$$p(z) = p(-z) \;\; \forall\, z \in \mathbb{Z}^d$$

otherwise it is **asymmetric**.

The random walk is **irreducible** if $\forall\, z \in \mathbb{Z}^d$ there exists $n \geq 0$

$$P(X_n = z) > 0.$$

The simple random walk is irreducible if and only if $p(z) > 0$ for all $z \in \{\pm e_j : 1 \leq j \leq d\}$.

Basic asymptotic results for simple random walks.

Recurrence/Transience: The random walk is **recurrent** if

$$P(X_n = z \text{ infinitely often}) = 1 \ \forall z \in \mathbb{Z}^d,$$

otherwise it is **transient**.

Recurrent: symmetric random walks in dimensions d=1,2.

Transient: asymmetric random walks, symmetric random walks in dimensions $d \geq 3$.

`https://upload.wikimedia.org/wikipedia/commons/c/cb/Random_walk_25000.svg`

Law of Large Numbers:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_n = v := \sum_{z\in\mathbb{Z}^d} zp(z) \ \text{ almost surely}$$

Functional Central Limit Theorem:

$$\hat{B}_t^n := \frac{X_{\lfloor nt \rfloor} - vt}{n^{1/2}}\,,\ t \geq 0\,,$$

$$(\hat{B}_t^n)_{t\geq 0} \xrightarrow[n\to\infty]{D} (B_t)_{t\geq 0}$$

where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian Motion.

Local Central Limit Theorem:

$$P(X_n = z) \approx \frac{e^{-\frac{(x\cdot\Gamma^{-1}x)}{}}}{(2\pi n)^{\frac{d}{2}}\sqrt{\det\Gamma}}$$

Applications of random walks and its generalizations pop up everywhere:

key model in the fields of computer science, physics, chemistry, biology, economics, etc.

Computer science:

PageRank. It calculates the importance of web pages by walking randomly among them. Researchers have developed a series variants of Random Walk.

Network topology: algorithms based on random walks in the area of collaborative filtering, link prediction and recommender system. Compared with other alternative approaches, random walk based algorithms can incorporate a great deal of contextual information.

Additionally computer vision, semi-supervised learning, complex social network analysis

Denoise probability models?

The Self Repelling Random Walk

Joint work with Thomas Mountford (EPFL) and Leandro Pimentel (UFRJ) 2014

Fix a non-decreasing function $w : \mathbb{Z} \to \mathbb{R}_+$ such that

$$\lim_{z \to \infty} \big(w(z) - w(-z) \big) > 0 \,.$$

$(X(k))_{k \geq 0}$ starting at $X(0) = 0$ is a **Self-repelling random walk (SRRW)** if

$$P\big(X(k+1) = X(k) \pm 1 \,\big|\, X(0), \dots, X(k)\big)$$

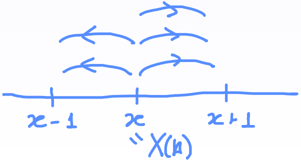
equal to

$$\frac{w\big(\mp\big(l^+(k, X(k)) - l^-(k, X(k))\big)\big)}{w\big(l^+(k, X(k)) - l^-(k, X(k))\big) + w\big(l^-(k, X(k)) - l^+(k, X(k))\big)}.$$

where

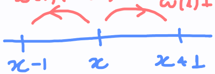
$$l^{\pm}(k, x) = \#\big\{0 \le j \le k-1 :\; X(j) = x,\; X(j+1) = x \pm 1\big\},$$

$l^-(k,x) = 2$ $\quad l^+(k,x) = 3$



$l^+(k,x) - l^-(k,x) = 1$

$q = \dfrac{w(-1)}{w(1)+w(-1)}$ $\quad p = \dfrac{w(1)}{w(1)+w(-1)}$



Tóth and B. Vetõ (2008) conjectured that $X(k)/\sqrt{k}$ converges in distribution to the uniform distribution on $(-1, 1)$.

We proved this conjecture and our main result is the following:

Theorem 1

Let $(X(k))_{k\geq 0}$ be the SRRW as described above. We have that as $k \to \infty$, $\frac{X(k)}{\sqrt{k}}$ converges in distribution to the uniform distribution on $(-1, 1)$.

Remark: Dumaz and Tóth (2013) obtain an analogous result is shown for the self repelling random walk with undirected edges. It is also worth mention that similar questions arise for random walks with site repulsion, Tóth and B. Vetõ (2011).

The previous Theorem is a straightforward consequence of the following local central limit theorem for the self-repelling random walk $(X(k))_{k\geq 0}$:

Theorem 2

There exists $1/2 < \alpha < 1$ *such that, for every* $\epsilon > 0$*, we can take* $k_0 = k_0(\epsilon)$ *sufficiently large so that if* $k \geq k_0$ *then*

$$P(X_k = x) \geq \frac{1-\epsilon}{\sqrt{k}},$$

for every $|x| \leq \sqrt{k} - k^{\frac{\alpha}{2}}$ *with the same parity as* k*.*

The definition of the SRRW leads us naturally to a Ray-Knight approach in order to obtain results for the SRRW.

The main tool is a representation of the local times on the inverse local times,

$$T^{\pm}_{x,m} = \min\left\{k \geq 0 : l^{\pm}(k,x) = m\right\},$$

in terms of independent ergodic Markov chains.

To simplify the notation suppose that:
(1) $k = n^2$ even though, obviously a typical positive integer is not a perfect square (Term n should be thought of as the integer part of $\sqrt{k}$);
(2) x and n^2 are even (note that $(X(k))_{k\geq 0}$ has period 2);
(3) $x \leq 0$ (the self-repelling walk is symmetric).

Note that $P\big(X_{n^2} = x\big)$ is equal to

$$P\big(\exists\, 0 \leq m \leq n^2 \text{ such that } T^{+}_{x-1,m} = n^2\big) + P\big(\exists\, 0 \leq m \leq n^2 \text{ such that } T^{-}_{x+1,m} = n^2\big)$$
$$= \sum_{m=0}^{n^2} P\big(T^{+}_{x-1,m} = n^2\big) + \sum_{m=0}^{n^2} P\big(T^{-}_{x+1,m} = n^2\big)\,. \quad (1)$$

Our first step is to consider for which values of m the contribution of $P(T^+_{x-1,m} = n^2)$ is relevant in the sum above. We claim that m should be $(n - |x|)/2$ plus a term of order $\sqrt{n}$, otherwise the contribution of $P(T^+_{x-1,m} = n^2)$ can be neglected. Indeed this is the content of the Lemma 3 which also aims at providing precise asymptotics for $P(T^+_{x-1,m} = n^2)$ for the right m. Before we state the result we need to fix some notation. Recall that σ^2 is the variance of the stationary distribution ν. Also define

$$\theta_u(v) := \frac{u}{2}\Big(1 - \frac{|v|}{u}\Big)\,,\ u > 0\,,\ v \in \mathbb{R}\,.$$

Lemma 3

There exists $1/2 < \alpha < 1$ such that, for every $\varepsilon > 0$ and $K > 0$, there exists $n_0 = n_0(\epsilon, K)$ sufficiently large such that

$$\sqrt{\beta_n \pi}\, n^{3/2} P\big(T^{\pm}_{x,\theta_n(x)+c\sqrt{n}} = n^2\big) \;\geq\; e^{-\frac{4\,c^2}{\beta_n}} \;-\; \varepsilon\,,$$

for all $n \geq n_0$, $|x| \leq n - n^{\alpha}$ with the same parity as n^2 and $c \in \big\{\tilde{c} \in (-K, K) : \theta_n(x) + \tilde{c}\sqrt{n} \in \mathbb{N}\big\}$, where

$$\beta_n = \frac{2\sigma^2\big((1+\frac{|x|}{n})^3 + (1-\frac{|x|}{n})^3\big)}{3}\,.$$

Tree Builder Random Walk

Joint work with Giulio Iacobelli (UFRJ), Leonel Zuaznabar (UFABC), Rodrigo Ribeiro (PUC-Chile) 2022

If T is a tree, $V(T)$ and $E(T)$ are its vertex and edge sets.

Ω collection of pairs (T, x), where T is a tree and $x \in V(T)$ is one of its vertices.

Fix

T_0 locally finite tree

s positive integer

$\boldsymbol{\xi} = \{\xi_n\}_{n\in\mathbb{N}}$ sequence of non-negative integer random variables

The **TBRW** is a stochastic processes $\{(T_n, X_n)\}_{n\geq 0}$ on Ω defined according to the following update rules:

1. Obtain T_{n+1} from T_n as follows:
   if $n = 0 \mod s$, add ξ_n new leaves to X_n,
   if $n \neq 0 \mod s$, $T_{n+1} = T_n$.

2. Choose uniformly one edge in $\{\{X_n, y\} : \{X_n, y\} \in E(T_{n+1})\}$, i.e., an edge incident to X_n in T_{n+1}, and set X_{n+1} as the chosen neighbor of X_n.

If $\boldsymbol{\xi}$ is a sequence of independent random variables, the **TBRW** process is a Markov chain.

Denote by $P_{T_0, x_0, s, \xi}(\cdot)$ the law of $\{(T_n, X_n)\}_{n\in\mathbb{N}}$ when $(T_0, X_0) = (T_0, x_0)$, and by $\mathbb{E}_{T_0, x_0, s, \xi}(\cdot)$ the corresponding expectation.

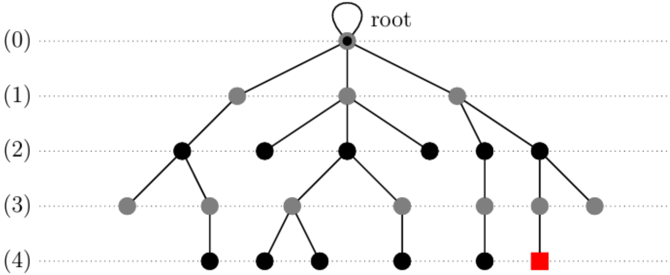


FIGURE 2. If the random walk X at an even time is in the squared vertex, whose level is (4), then the walker X is even and new leaf can only be added to vertices with even level, unless the walker uses the self-loop at the root.

$$\inf_{n\in\mathbb{N}} \mathrm{P}\left(\xi_n \geq 1\right) = \kappa > 0. \tag{UE}$$

$$\sup_{n\in\mathbb{N}} \mathrm{E}(\xi_n^r) \leq M < \infty. \tag{M-r}$$

For $S_n := \sum_{j=1}^{n} \xi_j$, we say $\boldsymbol{\xi}$ satisfies assumption (S) if there exists a positive constant c and a function $g : \mathbb{N} \setminus \{0\} \to \mathbb{R}^+$ of non-summable inverse ($\sum_{n=1}^{\infty} \frac{1}{g(n)} = \infty$) such that

$$P\left(\limsup_{n\to\infty} \frac{S_n}{g(n)} \leq c\right) = 1; \tag{S}$$

$\boldsymbol{\xi}$ satisfies condition (I) if there exist a positive constant c and a positive function $f : \mathbb{N} \setminus \{0\} \to \mathbb{R}^+$ of summable inverse ($\sum_{n=1}^{\infty} \frac{1}{f(n)} < \infty$), such that

$$P\left(\liminf_{n\to\infty} \frac{S_n}{f(n)} \geq c\right) = 1. \tag{I}$$

Theorem 4 (Recurrence/Traps for s even)

Consider a (s,ξ)-TBRW process with s even. For every initial state (T_0, x_0) with T_0 finite, there exist two regimes:

(i) (Recurrence is inherited) *if ξ satisfies condition (S), then the TBRW is recurrent.*

(ii) (The dangerous environment) *if ξ is an independent environment satisfying condition (I), then there exists n such that the walker gets trapped at time n, $P_{T_0,x_0,s,\xi}$-almost surely, i.e.*

$$P_{T_0,x_0,s,\xi}\left(\exists\, x \in \cup_n V(T_n) \text{ and } k \text{ such that } X_{sn+k} = x\ \forall\, n\right) = 1.$$

Theorem 5 (Null recurrence for s even)

Consider a $(s,\xi) - TBRW$ process with s even and independent environment ξ satisfying conditions (S), (UE) and $(M1)$. Then the TBRW is null recurrent.

Theorem 6 (Ballisticity for s odd)

If s is odd and $\boldsymbol{\xi}$ *is an independent environment satisfying* (UE) *and (M-r), then the TBRW is ballistic, i.e*

$$\liminf_{n\to\infty} \frac{d(X_n, root)}{n} > 0$$

almost surely.

In a recent paper Rodrigo Ribeiro proved a LLN and CLT for the TBRW with s odd.

Excited Random Walks

Joint work with Giulio Iacobelli (UFRJ) and Rodrigo Alves (PUC-RJ)

Excited random walk (ERW) is a model introduced by Benjamini and Wilson 2003.

It's a discrete time RW (X_n) in $\mathbb{Z}^d$, $d \geq 2$ with $X_0 = 0$ such that

- For $\delta \in (1/2, 1]$ fixed, at the first visit to a site, it will jump in the following way:

$$P(X_{n+1} = X_n + e_1 | X_0, ..., X_n) = \delta / d$$
$$P(X_{n+1} = X_n - e_1 | X_0, ..., X_n) = (1 - \delta)/d$$

  and $\forall i \in \{2, 3, \ldots, d\}$

$$P(X_{n+1} = X_n \pm e_i | X_0, ..., X_n) = 1/2d$$

- On an already visited site, the RW jumps to any nearest neighbor with uniform probability.

Benjamini and Wilson 2003 proved that ERW in $\mathbb{Z}^d$, $d \geq 2$ is transient to the right

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty \qquad \text{a.s..}$$

Furthermore, they also show that, if $d \geq 4$, ERW is ballistic to the right.

$$\liminf_{n\to\infty} \frac{X_n \cdot e_1}{n} > 0 \qquad \text{a.s..}$$

- Kozma 2003 and 2005 extended the proof of ballisticity for ERW to $d = 3$ and $d = 2$, respectively.
- Bérnard and Ramirez 2007 proved a Law of Large Numbers and a Central Limit Theorem for ERW with $d \geq 2$.

A more robust technique was developed by Menshikov, Popov, Ramirez and Vachkovskaia 2012. They also considered a more general model.

- on already visited sites the process behaves like a *d*-dimensional martingale with bounded jumps (rather than a SSRW),
- on the first time a site is visited the process has bounded jumps, satisfies UEC and drift condition in an arbitrary direction ℓ.

They call this model *generalized excited random walk* (GERW) and they showed that GERW with a drift condition in direction ℓ, is ballistic in that direction.

$d \geq 2$

$\{\xi_i\}_{i \geq 1}$ be the increments of a d-martingale with values in $\mathbb{Z}^d$ and zero-mean vector

$\{\gamma_i\}_{i \geq 1}$ be a sequence of $\mathbb{Z}^d$ random vectors

ℓ be a direction in $\mathbb{S}^{d-1}$ which is the unit sphere of $\mathbb{R}^d$.

Condition I There exists a positive constant K such that

$$\sup_{n \geq 1} ||\xi_n|| \leq K \quad \text{and} \quad \sup_{n \geq 1} ||\gamma_n|| \leq K \ ,$$

on every realization.

Condition II For every $n \geq 1$, we have that

$$E[\xi_n | \mathcal{F}_{n-1}] = 0 \quad \text{and} \quad E[\gamma_n \cdot \ell | \mathcal{F}_{n-1}] \geq \lambda \ ,$$

where λ is a positive constant.

$\{U_i\}_{i\geq 1}$ sequence of IID random variables with uniform distribution in $[0,1]$ independent of the $\{\xi_i\}_{i\geq 1}$ and $\{\gamma_i\}_{i\geq 1}$.

$\{p_n\}_{n\geq 1}$ be a sequence such that $p_n \in (0,1]\ \forall\, n \geq 1$.

Definition of the p_n-GERW $X = \{X_n\}_{n\geq 0}$.

$X_0 = 0$ and

$$X_n := \sum_{i=1}^{n} \left(1_{E_{i-1}} \xi_i + 1_{E_{i-1}^c \cap \{U_i > p_i\}} \xi_i + 1_{E_{i-1}^c \cap \{U_i \leq p_i\}} \gamma_i \right), \ \ n \geq 1\,,$$

where $E_0 := \emptyset$ and, for $i \geq 1$, $E_i := \{\exists\ k < i \ \text{ such that } \ X_k = X_i\}$.

A special case: p_n**-ERW.** $\{\xi_i\}_{i\geq 1}$ is IID with zero-mean vector and finite covariance matrix. Additionally, we assume $P[\xi_i \cdot e_k = 0] < 1$ for all $i \geq 0$ and for each $k \in \{1, 2, \ldots, d\}$. The sequence $\{\gamma_i\}_{i\geq 1}$ is also IID and the sequences $\{\xi_i\}_{i\geq 1}$ and $\{\gamma_i\}_{i\geq 1}$ are independent.

We suppose $p_n = \mathcal{C} n^{-\beta} \wedge 1$, with $\beta > 1/2$ and $\mathcal{C} > 0$.

In some cases we also relax condition I to

Condition I* For all $k \geq 1$ and $\theta < \beta - 1/2$, where $\beta > 1/2$, we have

$$\sup_{k\geq 1} \frac{E[\|\xi_k\|]}{k^\theta} < \infty \quad \text{and} \quad \sup_{k\geq 1} \frac{E[\|\gamma_k\|]}{k^\theta} < \infty \, .$$

If a process X satisfies Condition I* and Condition II, we call X a p_n-GERW*.

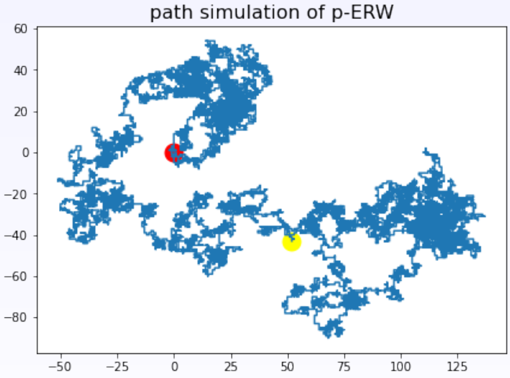


Figure: 20000 steps simulation of p-ERW for $p = 0.03$. $X_{20000} = (52, -43)$

We obtain $\overline{v \cdot e_1} = 0.002087$.

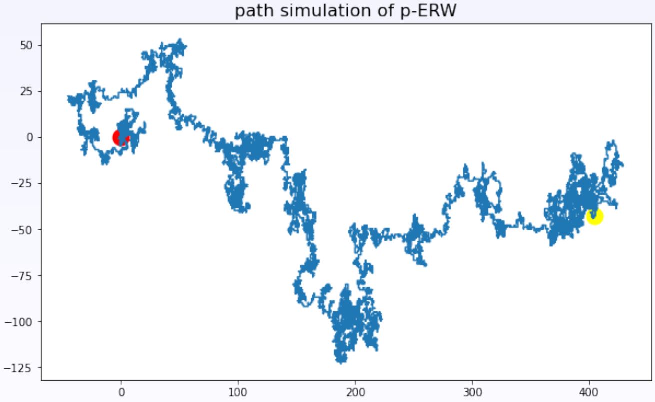


Figure: 20000 steps simulation of p-ERW for $p = 0.25$. $X_{20000} = (404, -43)$.

We obtain $\overline{v \cdot e_1} = 0.019856$.

Theorem 7

Let X be a p_n-GERW in direction ℓ with $d \geq 2$ and $\beta > 1/2$. Suppose that*

$$\lim_{k\to\infty} k^{-1/2} E\Big[\sup_{1\leq i\leq k} \|\xi_i\|\Big] = 0\,,$$

and $\exists\ C = ((c_{i,j}))$ continuous $d \times d$ matrix-valued function on $[0,\infty)$ satisfying $C(0) = 0$ and

$$\sum_{i,j=1}^{d} (c_{i,j}(t) - c_{i,j}(s))\alpha_i\alpha_j \geq 0 \quad \text{for any } \alpha \in \mathbb{R}^d, \quad t > s \geq 0\,,$$

such that

$$\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor} \xi_i\xi_i^T \xrightarrow[n\to\infty]{} C(t) \ \ \textit{in probability}\,.$$

Then

$$\hat{B}_t^n := \frac{X_{\lfloor nt\rfloor}}{n^{1/2}} + (nt - \lfloor nt\rfloor)\frac{(X_{\lfloor nt\rfloor+1} - X_{\lfloor nt\rfloor})}{n^{1/2}}\,,\ t \geq 0\,,$$

converges in distribution to a process with independent Gaussian increments with sample paths in $C_{\mathbb{R}^d}[0,\infty)$.

Theorem 8

Let X be a p_n-ERW in direction ℓ with $d = 2$, $p_n = \mathcal{C}n^{-1/2} \wedge 1$. Then $\{\hat{B}^n_\cdot\}_{n \geq 1}$ converges in distribution to a 2-dimensional Brownian Motion.

π_d denotes the probability that the d-dimensional random walk with increments $\{\xi_i\}_{i \geq 1}$ never returns to the origin.

Theorem 9

Let X be a p_n-ERW in direction $\ell_{\mathbb{D}}$ with $d \geq 4$, $p_n = \mathcal{C}n^{-1/2} \wedge 1$. Then $\{\hat{B}^n_\cdot\}_{n \geq 1}$ is tight and there exists a Brownian Motion $W_\cdot$ such that for every limit point $Y_\cdot$ of $\{\hat{B}^n_\cdot\}_{n \geq 1}$

$$\left\{ W_t \cdot \ell_{\mathbb{D}} + 2c_1\sqrt{t} \right\}_{t \geq 0} \preceq \{ Y_t \cdot \ell_{\mathbb{D}} \}_{t \geq 0} \preceq \left\{ W_t \cdot \ell_{\mathbb{D}} + 2c_2\sqrt{t} \right\}_{t \geq 0},$$

where $c_1 = \mu_\gamma(1 - \sqrt{1 - \pi_{d-k}})$, $c_2 = \mu_\gamma\sqrt{\pi_d}$ with $\mu_\gamma := E[\gamma_i \cdot \ell_{\mathbb{D}}]$.

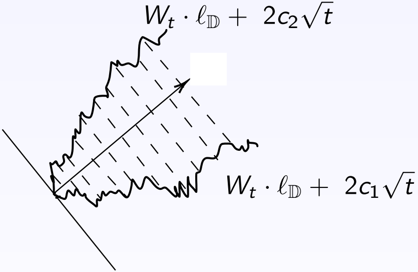


Figure: "Cone" region representation around the direction $\ell_{\mathbb{D}}$.

Conjecture: Let X be a p_n-GERW in direction $\ell \in \mathbb{S}^{d-1}$ with $d \geq 3$, $p_n = \mathcal{C}n^{-1/2} \wedge 1$. Then $\{\hat{B}^n_\cdot\}_{n \geq 1}$ converges in distribution to

$$\{W_t \cdot \ell + 2\mu_\gamma \sqrt{\pi_d\, t}\}_{t>0},$$

where $W_\cdot$ is a Brownian Motion.

Theorem 10

Let X be a p_n-GERW in direction ℓ with $d \geq 2$, $p_n = \mathcal{C}n^{-1/2} \wedge 1$. Let $\mathcal{R}_n^X$ be the range of X up to time n. Then, for $\delta > \pi_d$

$$P[\exists n_\delta \text{ such that } \forall n \geq n_\delta : \; |\mathcal{R}_n^X| \leq \delta n] = 1\,.$$

$\pi_d = 0$ for $d = 2$, whereas for $d \geq 3$, $\pi_d \in (0, 1]$.

Conjecture: Let X be a p_n-ERW in direction $\ell \in \mathbb{S}^{d-1}$ with $d \geq 3$, $p_n = \mathcal{C} n^{-\beta} \wedge 1$, with $\beta \geq 1/2$. Let $\mathcal{R}_n^X$ be the range of the process up to time n. Then we have

$$\frac{|\mathcal{R}_n^X|}{n} \xrightarrow[n \to \infty]{} \pi_d \quad \text{a.s.}.$$

If the conjecture holds true, we would be able to extend the result in Theorem 9 to $d = 3$ and to any direction in the unit sphere. Moreover Theorem 9 will hold with $c_1 = \mu_\gamma(1 - \sqrt{1 - \pi_d})$. Note, however, that this is not yet enough to imply the convergence.

Table: Summary of the results for p_n-GERW.

p_n-GERW* ($\beta > 1/2$, $d \geq 2$)	*Convergence in distribution to a Gaussian Process.*
p_n-ERW ($\beta = 1/2$, $d = 2$)	*Convergence in distribution to a Brownian Motion.*
p_n-ERW ($\beta = 1/2$, $d \geq 4$)	*All sub-sequences converge, in distribution, to a process which is stochastically dominated in the drift direction below and above by a Brownian Motion plus a continuous function.*
p_n-GERW (β small, $d \geq 2$)	*Directional transience.*